Introduction to Elementary Particles
Instructor's Solution Manual

29th August 2008

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http://academic.reed.edu/physics/faculty/griffiths.html

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1
Historical Introduction to the Elementary Particles

Problem 1.1

For an undeflected charged particle, $qE = qvB \Longrightarrow \boxed{v = \frac{E}{B}}.$

With just a magnetic field, $qvB = m\frac{v^2}{R} \Longrightarrow \frac{q}{m} = \frac{v}{BR} = \boxed{\frac{E}{B^2R}}.$

Problem 1.2

$r_0 = 10^{-15}$ m; $\hbar = 6.58 \times 10^{-22}$ MeV s; $c = 3.00 \times 10^8$ m/s;

so $m = \frac{\hbar}{2r_0c} = \left(\frac{\hbar c}{2r_0}\right)\frac{1}{c^2} = \boxed{98.7\ \text{MeV}/c^2}.$

Observed $m_\pi = 138$ MeV$/c^2$. $\boxed{\text{Off by a factor of 1.4.}}$

Problem 1.3

$r_0 = 10^{-15}$ m; $\hbar = 6.58 \times 10^{-22}$ MeV s; $c = 3.00 \times 10^8$ m/s;

$m_e = 0.511$ MeV$/c^2$.

$\Delta x \Delta p \geq \frac{\hbar}{2}$ so $p_{\text{min}} = \frac{\hbar}{2r_0} = \left(\frac{\hbar c}{2r_0}\right)\frac{1}{c} = \boxed{98.7\ \text{MeV/c}}.$

$E_{\text{min}} = \sqrt{p_{\text{min}}^2 c^2 + m_e^2 c^4} = \boxed{98.7\ \text{MeV}}.$

The energy of an electron emitted in the beta decay of tritium is < 17 keV.

Problem 1.4

$$m_\Lambda = \frac{1}{3}\left[2\left(m_N + m_\Xi\right) - m_\Sigma\right].$$

$$m_N = 938.9; \quad m_\Xi = 1318.1; \quad m_\Sigma = 1190.5.$$

$$\text{So } m_\Lambda = \frac{1}{3}\left[2\left(2257.0\right) - 1190.5\right] = \boxed{1107.8\ \text{MeV}/c^2.}$$

Observed $m_\Lambda = 1115.7\ \text{MeV}/c^2$. $\boxed{\text{Off by } 0.7\%.}$

Problem 1.5

$$m_\eta^2 = \frac{1}{3}\left[2\left(m_K^2 + m_K^2\right) - m_\pi^2\right] = \frac{1}{3}\left(4m_K^2 - m_\pi^2\right).$$

$$m_K = 495.67; \quad m_\pi = 138.04.$$

$$m_\eta^2 = \frac{1}{3}\left[9.637 \times 10^5\right] = 3.212 \times 10^5 \Rightarrow \boxed{m_\eta = 566.8\ \text{MeV}/c^2.}$$

Actually $m_\eta = 547.3\ \text{MeV}/c^2$. $\boxed{\text{Off by } 3.5\%.}$

Problem 1.6

$$M_\Delta - M_{\Sigma^*} = 1232 - 1385 = -153$$

$$M_{\Sigma^*} - M_{\Xi^*} = 1385 - 1533 = -148. \quad \textit{Average}: -151.$$

$$\therefore M_\Omega = M_{\Xi^*} + 151 = 1533 + 151 = \boxed{1684\ \text{MeV}/c^2.}$$

Actually $M_\Omega = 1672\ \text{MeV}/c^2$. $\boxed{\text{Off by } 0.7\%.}$

Problem 1.7

(a) $\boxed{\Delta^- \longrightarrow n + \pi^- \text{ or } \Sigma^- + K^0}$

$\boxed{\Sigma^{*+} \longrightarrow p + \bar{K}^0; \quad \Sigma^+ + \pi^0; \quad \Sigma^+ + \eta; \quad \Sigma^0 + \pi^+; \quad \Lambda + \pi^+; \quad \Xi^0 + K^+}$

$\boxed{\Xi^{*-} \longrightarrow \Sigma^0 + K^-; \quad \Sigma^- + \bar{K}^0; \quad \Lambda + K^-; \quad \Xi^0 + \pi^-; \quad \Xi^- + \pi^0; \quad \Xi^- + \eta}$

(b) Kinematically allowed:

$\boxed{\Delta^- \longrightarrow n + \pi^-}$

$\boxed{\Sigma^{*+} \longrightarrow \Sigma^+ + \pi^0; \quad \Sigma^0 + \pi^+; \quad \Lambda + \pi^+}$

$\boxed{\Xi^{*-} \longrightarrow \Xi^0 + \pi^-; \quad \Xi^- + \pi^0}$

Problem 1.8

(a) With a strangeness of -3, the Ω^- would have to go to $(\Xi^0 + K^-)$ or $\left(\Xi^- + \bar{K^0}\right)$ to conserve S and Q. But the ΞK combination is too heavy (at least 1808 MeV/c^2, whereas the Ω^- is predicted – see Problem 1.6 – to have a mass of only 1684).

(b) About $\boxed{0.5\text{ cm}}$; $t = d/v = (5 \times 10^{-3}\text{ m})/(3 \times 10^7\text{ m/s}) = \boxed{2 \times 10^{-10}\text{ s.}}$ (Actually, $t = 0.8 \times 10^{-10}$ s.)

Problem 1.9

$$\left.\begin{array}{l} \Sigma^+ - \Sigma^- = 1189.4 - 1197.4 = -8.0 \\ p - n + \Xi^0 - \Xi^- = 938.3 - 939.6 + 1314.8 - 1321.3 = -7.8 \end{array}\right\} \boxed{3\% \text{ difference.}}$$

Problem 1.10

Roos lists a total of 30 meson types; in the first column is the particle name at the time, in the second column the quoted mass (in MeV/c^2), and in the third its current status.

meson	mass	status	exotic	meson	mass	status	exotic
π	138	π		ρ	755	$\rho(770)$	
K	496	K		ρ_2	780	$f_0(600)?$	
K_3	1630	dead	yes	ρ_1	720	$f_0(600)?$	
χ_2	1340	$f_0(1370)?$		ψ_4	760	dead	yes
κ_3	1275	$f_1(1285)?$		K_1^{**}	730	dead	
K^{**}	1260	$K_1(1270)?$		δ	645	dead	
f	1253	$f_2(1270)?$		α	625	dead	
K_5^*	1150	dead	yes	ψ_3	597	dead	yes
χ_1	1045	$a_0(980)?$		ζ	556	dead	
κ_2	1040	dead		η	549	η	
κ_1	1020	$\phi(1020)$		φ_2	520	dead	
ψ_5	990	dead	yes	ψ_2	440	dead	yes
K_1^*	888	$K^*(892)$		φ_1	395	dead	
φ_3	885	$\eta'(985)?$		ψ_1	330	dead	yes
ω	781	$\omega(782)$		ω_{ABC}	317	dead	

Problem 1.11

From the last column in Problem 1.10 I count 7 exotic species, all of them now dead. Of the surviving particles (of course) none is exotic.

Problem 1.12

1 quark (u): *one* ($u\bar{u}$); 2 quarks (u, d): *four* ($u\bar{u}$, $u\bar{d}$, $d\bar{u}$, $d\bar{d}$);
3 quarks (u, d, s): *nine*; 4 quarks (u, d, s, c): *sixteen*;
5 quarks (u, d, s, c, b): *twenty-five*; 6 quarks (u, d, s, c, b, t): *thirty-six*.
The general formula for n flavors is $\boxed{n^2.}$

Problem 1.13

1 quark (u) $\Longrightarrow$ *1* baryon (uuu);
2 quarks (u, d) $\Longrightarrow$ *4* baryons (uuu, uud, udd, ddd);
3 quarks (u, d, s) $\Longrightarrow$ *10* baryons (baryon decuplet).

For n quarks, we can have

$$\begin{aligned}&\text{all three quarks the same}: \quad n \text{ ways}\\&\text{two the same, one different}: \quad n(n-1) \text{ ways}\\&\text{all three different}: \quad n(n-1)(n-2)/6 \text{ ways.}\end{aligned}$$

[For the third type of combination, divide by six to cover the equivalent permutations ($uds = usd = dus = dsu = sud = sdu$).]
So the total is

$$\begin{aligned}&n + n(n-1) + n(n-1)(n-2)/6\\=&n + n^2 - n + n(n-1)(n-2)/6\\=&\frac{n}{6}\left[6n + (n-1)(n-2)\right]\\=&\frac{n}{6}\left(6n + n^2 - 3n + 2\right)\\=&\frac{n}{6}\left(n^2 + 3n + 2\right)\\=&\boxed{\frac{n(n+1)(n+2)}{6}.}\end{aligned}$$

Thus for four quarks we have *20* baryon types, for five quarks, *35*, and for six quarks, *56*.

Problem 1.14

uuu	uuc	ucc	$\underline{ccc}$
uud	udc	dcc	1 has C = 3
udd	ddc	$\underline{scc}$	
ddd	usc	3 have C = 2	
uus	dsc		
uds	$\underline{ssc}$		
dds	6 have C = 1		
uss			
dss			
$\underline{sss}$			
10 have C = 0			

Problem 1.15

$u\bar{u}$	$c\bar{u}$	$u\bar{c}$	$c\bar{c}$
$u\bar{d}$	$c\bar{s}$	$d\bar{c}$	
$d\bar{u}$	$c\bar{d}$	$s\bar{c}$	
$d\bar{d}$	3 have C = 1	3 have C = −1	
$u\bar{s}$			
$s\bar{u}$			
$d\bar{s}$			
$s\bar{d}$			
$s\bar{s}$			
10 have C = 0 (including $c\bar{c}$)			

Problem 1.16

$q\bar{q}$	meson	mass	year
$u\bar{u}$	π^0 (*)	134.98	1950
$u\bar{d}$	π^+	139.57	1947
$d\bar{d}$	π^0 (*)	134.98	1950
$u\bar{s}$	K^+	493.68	1949
$d\bar{s}$	K^0	497.65	1947
$s\bar{s}$	η (*)	547.51	1962
$c\bar{u}$	D^0	1864.5	1976
$c\bar{d}$	D^+	1869.3	1976

$q\bar{q}$	meson	mass	year
$c\bar{s}$	D_s^+	1968.2	1977
$c\bar{c}$	$\eta_c(1S)$	2980.4	1980
$u\bar{b}$	B^+	5279.0	1983
$d\bar{b}$	B^0	5279.4	1983
$s\bar{b}$	B_s^0	5367.5	1993
$c\bar{b}$	B_c^+	6286	1998
$b\bar{b}$	$\Upsilon(1S)$	9460.3	1977

All masses are in MeV/c^2; (*) indicates that the particle is a combination of different quark states.

qqq	baryon	mass	year
uuu	Δ^{++}	1232	1951
uud	p	938.27	1911
udd	n	939.57	1932
ddd	Δ^-	1232	1951
uus	Σ^+	1189.37	1963
uds	Λ	1115.68	1950
dds	Σ^-	1197.45	1965
uss	Ξ^0	1314.83	1963
dss	Ξ^-	1321.31	1963
sss	Ω^-	1672.45	1964

qqq	baryon	mass	year
uuc	Σ_c^{++}	2454.02	1975
udc	Λ_c^+	2286.46	1975
ddc	Σ_c^0	2453.76	1975
usc	Ξ_c^+	2467.9	?
dsc	Ξ_c^0	2471.0	?
ssc	Ω_c^0	2697.5	?
ucc	Ξ_{cc}^{++}		
dcc	Ξ_{cc}^+	3518.9	2002
scc	Ω_{cc}^+		
ccc	Ω_{ccc}^{++}		

qqq	baryon	mass	year
uub	Σ_b^+	5807.8	2007
udb	Λ_b^0	5624	?
ddb	Σ_b^-	5815.2	2007
usb	Ξ_b^0	5792	1995
dsb	Ξ_b^-	5792.9	2007
ssb	Ω_b^-		
ucb	Ξ_{cb}^+		
dcb	Ξ_{cb}^0		
scb	Ω_{cb}^0		
ccb	Ω_{ccb}^+		

qqq	baryon	mass	year
ubb	Ξ_{bb}^0		
dbb	Ξ_{bb}^-		
sbb	Ω_{bb}^-		
cbb	Ω_{ccb}^0		
bbb	Ω_{bbb}^-		

Blank spaces indicate that the particle has not yet been found (2008).

Problem 1.17

$$N: \quad 3\times 336 - 62 = \boxed{946}; \quad \text{actually } 939; \quad \text{error: } \boxed{0.7\%.}$$

$$\Sigma: \quad 2\times 336 + 540 - 62 = \boxed{1150}; \quad \text{actually } 1193; \quad \text{error: } \boxed{-3.6\%.}$$

$$\Lambda: \quad 2\times 336 + 540 - 62 = \boxed{1150}; \quad \text{actually } 1116; \quad \text{error: } \boxed{3.0\%.}$$

$$\Xi: \quad 336 + 2\times 540 - 62 = \boxed{1354}; \quad \text{actually } 1318; \quad \text{error: } \boxed{2.7\%.}$$

Problem 1.18

Quarks and leptons:

$$\boxed{\begin{array}{ll} ccc\ (q=-1):\ e^- & \bar{c}\bar{c}\bar{c}\ (q=1):\ e^+ \\ ccn\ (q=-\frac{2}{3}):\ \bar{u} & \bar{c}\bar{c}\bar{n}\ (q=\frac{2}{3}):\ u \\ cnn\ (q=-\frac{1}{3}):\ d & \bar{c}\bar{n}\bar{n}\ (q=\frac{1}{3}):\ \bar{d} \\ nnn\ (q=0):\ \bar{\nu}_e & \bar{n}\bar{n}\bar{n}\ (q=0):\ \nu_e \end{array}}$$

(The neutrinos could be switched, but this seems the most "natural" assignment.)

Mediators:

$$\boxed{\begin{array}{l} ccc\bar{n}\bar{n}\bar{n}\ (q=-1):\ W^- \\ nnn\bar{c}\bar{c}\bar{c}\ (q=+1):\ W^+ \\ \left.\begin{array}{r} ccc\bar{c}\bar{c}\bar{c}\ (q=0):\quad Z^0 \\ nnn\bar{n}\bar{n}\bar{n}\ (q=0):\quad \gamma \end{array}\right\}\ \text{or vice versa} \end{array}}$$

Gluons: We need matching triples of particles and antiparticles,

$cc n\bar{c}\bar{c}\bar{n}$ (3 different orderings for each triple, so 9 possibilities);
$cnn\bar{c}\bar{n}\bar{n}$ (3 different orderings for each triple, so 9 possibilities);
leading to a total of 18 possibilities.

Problem 1.19

There are at least four distinct answers, depending on the particle in question:

- Antiparticles (such as the positron) **annihilate** with the corresponding particle (the electron, in this case), and since there are lots of electrons in the lab, positrons don't stick around long enough to have any role in ordinary chemical processes. But if you could work in a total vacuum you could make atoms and molecules of antimatter, and all of chemistry would proceed just the same as with ordinary matter.
- Most elementary particles (such as muons, pions, and intermediate vector bosons) are intrinsically **unstable**; they disintegrate spontaneously in a tiny fraction of a second—not long enough to do any serious chemistry. You can make short-lived "exotic atoms", with (say) muons in orbit around the nucleus instead of electrons. Some of these systems last long enough to do spectroscopy.
- Neutrinos **interact** so **feebly** with matter that they have no impact on chemistry, even though we are in fact *bathed* in them all the time.
- Quarks are the basic constituents of protons and neutrons, so in an indirect way they *do* play a fundamental role in chemistry. (And gluons play a fundamental role in holding the nucleus together.) But because of **confinement**, they do not occur as free particles, only in composite structures, so they don't act as "individuals".

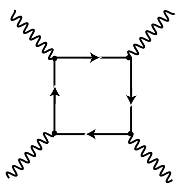
2
Elementary Particle Dynamics

Problem 2.1

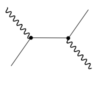
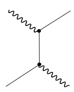
$$F_g = \frac{Gm^2}{r^2}; \qquad F_e = \frac{e^2}{4\pi\epsilon_0 r^2}$$

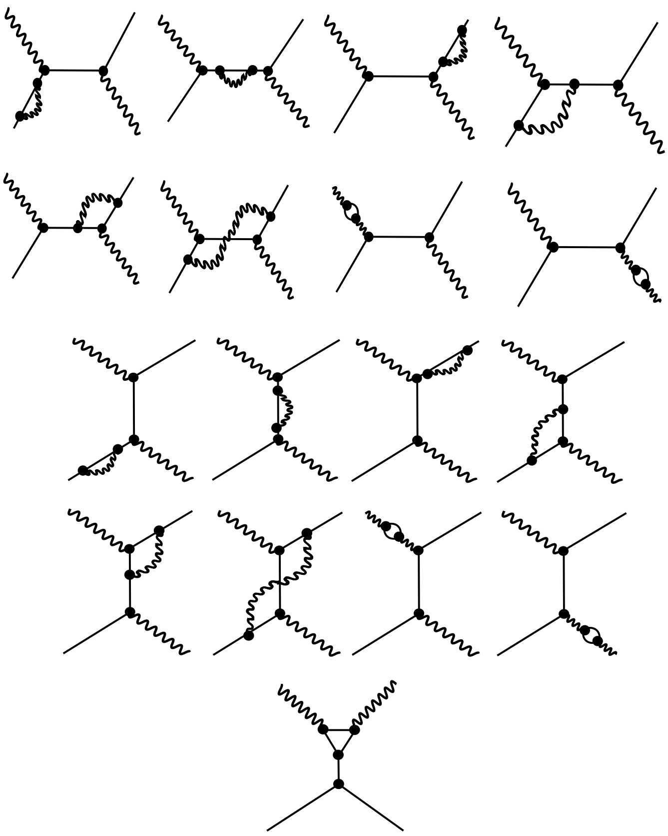
$$\frac{F_g}{F_e} = \frac{4\pi\epsilon_0 G m^2}{e^2} = \boxed{2.4 \times 10^{-43}}.$$

Problem 2.2

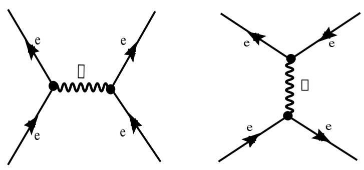


Problem 2.3

Eight are built on  ; eight are built on  ; one is special.



Problem 2.4



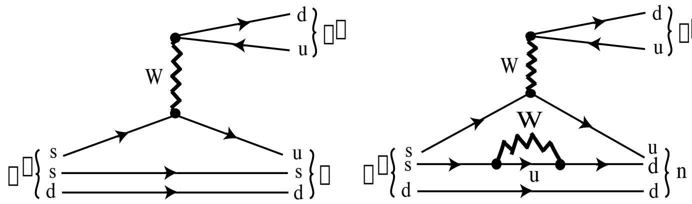
Energy and momentum are conserved at each vertex. Thus, for the virtual photon in the "horizontal" diagram

$$E = 2m_ec^2 \quad \text{and} \quad p = 0 \quad \Longrightarrow \quad \boxed{m = 2m_e \quad \text{and} \quad v = 0}$$

and in the "vertical" diagram,

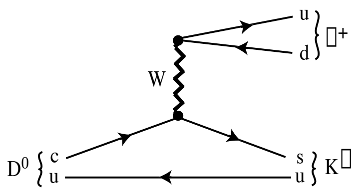
$$E = 0 \quad \text{and} \quad p = 0 \quad \Longrightarrow \quad \boxed{m = 0 \quad \text{and} \quad v = 0.}$$

Problem 2.5

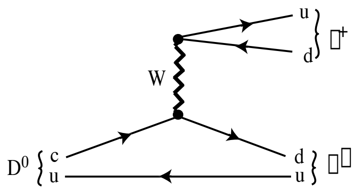


(a) $\Xi^- \longrightarrow n + \pi^-$ would be favored *kinematically*, but since *two s* quarks have to be converted, it requires an extra W^- (hence two extra weak vertices), and this makes it much less likely. $\boxed{\Lambda + \pi^- \text{ is favored.}}$ Experimentally, 99.887% go to $\Lambda + \pi^-$; the branching ratio for $n + \pi^-$ is less than 1.9×10^{-5}.

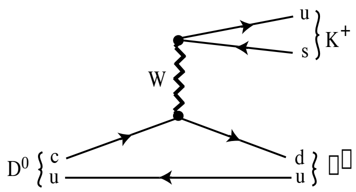
(b) $D^0 \longrightarrow K^- + \pi^+$: *Neither* vertex crosses generations:



$D^0 \longrightarrow \pi^- + \pi^+$: *One* vertex crosses generations:



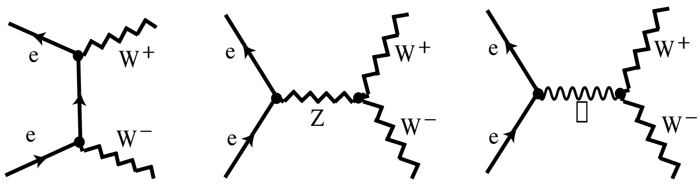
$D^0 \longrightarrow \pi^- + K^+$: *Both* vertices cross generations:



Because weak vertices within a generation are favored, $\boxed{K^- + \pi^+ \text{ is } \textit{most} \text{ likely, } K^+ + \pi^- \text{ least.}}$
Experimentally, the branching ratios are: 3.8% for $K^- + \pi^+$, 0.14% for $\pi^+ + \pi^-$, and 0.014% for $K^+ + \pi^-$.

(c) The b quark prefers to go to c ($V_{cb} = 0.042$, whereas $V_{ub} = 0.004$), so $\boxed{B \text{ should go to } D.}$

Problem 2.6



Problem 2.7

(a) Impossible (charge conservation)

(b) Possible, *electromagnetic*

(c) Impossible (energy conservation)

(d) Possible, *weak*

(e) Possible, *electromagnetic*

(f) Impossible (muon number conservation)

(g) Possible, *strong*

(h) Possible, *weak*

(i) Impossible (baryon number conservation)

(j) Possible, *strong*

(k) Impossible (baryon and lepton number conservation)

(l) Possible, *strong*

(m) Possible, *strong*

(n) Impossible (charge conservation)

(o) Possible, *weak*

(p) Impossible (charge conservation)

(q) Possible, *electromagnetic*

(r) Possible, *weak*

(s) Possible, *weak*

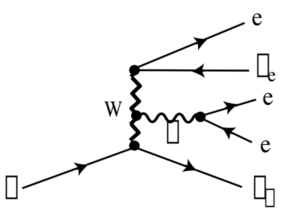
(t) Possible, *strong*

(u) Possible, *electromagnetic*

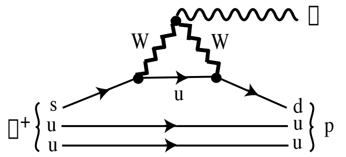
(v) Possible, *weak*

Problem 2.8

(a) $K^+ \longrightarrow \mu^+ + \nu_\mu + \gamma$; weak and electromagnetic interactions are involved:



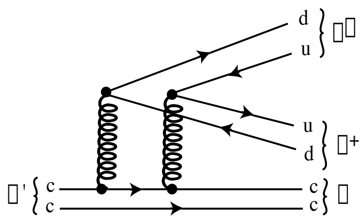
(b) $\Sigma^+ \longrightarrow p + \gamma$; weak and electromagnetic interactions are involved:



Problem 2.9

The lifetime tells us this is an OZI-suppressed strong interaction. Evidently the B meson must weigh *more than half* the Υ (just as the D weighs more than half the ψ). Thus the B meson should weigh $\boxed{\text{more than 4730 MeV}/c^2}$ (and it *does*).

Problem 2.10



Here's a typical contributing diagram. *Yes*, it is a strong interaction. *Yes*, it is OZI-suppressed. We should expect a lifetime *around* 10^{-20} seconds.

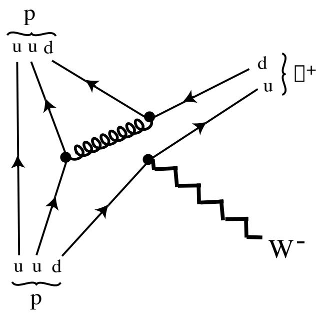
Problem 2.11

(a) Particle X has charge +1 and strangeness 0; it was presumably a proton.

(b) $K^- + p \longrightarrow K^0 + K^+ + \Omega^-$ (strong);
$\Omega^- \longrightarrow \Xi^0 + \pi^-$ (weak);
$\Xi^0 \longrightarrow \Lambda^0 + \pi^0$ (weak),
$\pi^0 \longrightarrow \gamma + \gamma$ (electromagnetic), and both photons undergo pair production $\gamma \longrightarrow e^+ + e^-$ (electromagnetic);
$\Lambda^0 \longrightarrow \pi^- + p$ (weak).

Problem 2.12

A pion seems most likely—it requires only one weak vertex, with no generation crossing, and it's light (hence kinematically favored):



3
Relativistic Kinematics

Problem 3.1

$$\left\{\begin{array}{ll} x' = \gamma(x - vt) & \Longrightarrow x'/\gamma = x - vt \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) & \Longrightarrow vt'/\gamma = vt - \frac{v^2}{c^2}x \end{array}\right\}$$

Adding these two equations:

$$\frac{1}{\gamma}(x' + vt') = x\left(1 - \frac{v^2}{c^2}\right) = x/\gamma^2 \qquad \therefore \boxed{x = \gamma(x' + vt').}$$

$$\left\{\begin{array}{ll} x' = \gamma(x - vt) & \Longrightarrow \frac{v}{c^2}\frac{x'}{\gamma} = \frac{v}{c^2}x - \frac{v^2}{c^2}t \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) & \Longrightarrow t'/\gamma = t - \frac{v}{c^2}x \end{array}\right\}$$

Adding these two equations:

$$\frac{1}{\gamma}\left(t' + \frac{v}{c^2}x'\right) = t\left(1 - \frac{v^2}{c^2}\right) = t/\gamma^2 \qquad \therefore \boxed{t = \gamma\left(t' + \frac{v}{c^2}x'\right).}$$

Also, $y = y'$, and $z = z'$. This confirms Eq. 3.3.

Problem 3.2

(a) From the Lorentz transformations (Eq. 3.1), $t' = \gamma\left(t - \frac{v}{c^2}x\right)$.

$$\therefore t_A{}' - t_B{}' = \gamma\left[t_A - t_B - \frac{v}{c^2}(x_A - x_B)\right].$$

If simultaneous in S (so that $t_A = t_B$), then $\boxed{t_A{}' = t_B{}' + \gamma\frac{v}{c^2}(x_B - x_A).}$

(b)

$$\gamma = \frac{1}{\sqrt{1 - \frac{9}{25}}} = \frac{5}{4}; \qquad x_B - x_A = 4 \text{ km} = 4 \times 10^3 \text{ m}.$$

$$t_A{}' = t_B{}' + \frac{\not{5}}{\not{4}} \cdot \frac{3}{\not{5}} \cdot \frac{\not{c}}{c^{\not{2}}} \left(\not{4} \times 10^3 \text{ m}\right) = t_B{}' + \frac{3 \times 10^3 \text{ m}}{3 \times 10^8 \text{ m/s}} = t_B{}' + 10^{-5} \text{ s}.$$

So $\boxed{\text{B went on first}}$; A came on $\boxed{10^{-5} \text{ seconds later.}}$

Problem 3.3

(a) The dimension along the direction of motion undergoes length contraction, but the other two dimensions do not. Thus, volumes transform according to $\boxed{V = V'/\gamma.}$

(b) The number of moleculues N is invariant. Density is $\rho = N/V$, so density transforms by $\boxed{\rho = N/V = \gamma N/V' = \gamma \rho'.}$

Problem 3.4

(a)

$$d = vt = \left(0.998 \times 3 \times 10^8 \text{ m/s}\right)\left(2.2 \times 10^{-6} \text{ s}\right) = \boxed{659 \text{ m}}. \quad \boxed{\text{No.}}$$

(b)

$$\gamma = \frac{1}{\sqrt{1 - (0.998)^2}} = 15.8.$$

$$d = v\,(\gamma t) = \gamma(659) = (15.8)(659) = \boxed{10{,}400 \text{ m}}. \quad \boxed{\text{Yes.}}$$

(c) $\boxed{\text{No.}}$ They only travel $10,400(2.6 \times 10^{-8})/(2.2 \times 10^{-6}) = 123$ m.

Problem 3.5

$d = 600$ m; $\tau = 2.2 \times 10^{-6}$ s.
The half-life $t_{1/2}$ is related to the lifetime τ by $t_{1/2} = (\ln 2)\tau$. Thus,

$$d = v(\gamma t_{1/2}) = \frac{v}{\sqrt{1 - (v/c)^2}}(\ln 2)\tau \Longrightarrow \boxed{v = 0.80\,c}$$

Problem 3.6

(a) Velocity of bullet relative to ground $= \frac{1}{2}c + \frac{1}{3}c = \frac{5}{6}c = \frac{10}{12}c$, but the getaway car goes $\frac{3}{4}c = \frac{9}{12}c$ (slower), so bullet $\boxed{\text{does}}$ reach target.

(b)

$$v = \frac{\frac{1}{2}c + \frac{1}{3}c}{1 + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\frac{5}{6}c}{\frac{7}{6}} = \frac{5}{7}c = \frac{20}{28}c,$$

which is *less* than $\frac{3}{4}c = \frac{21}{28}c$, so bullet $\boxed{\text{doesn't}}$ reach target.

Problem 3.7

$$M = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda M = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

Problem 3.8

From Eq. 3.8:

$$\begin{aligned}(x^{0\prime})^2 - (x^{1\prime})^2 &= \gamma^2(x^0 - \beta x^1)^2 - \gamma^2(x^1 - \beta x^0)^2 \\ &= \gamma^2 \Big[(x^0)^2 - \cancel{2\beta x^0 x^1} + \beta^2(x^1)^2 \\ &\quad -(x^1)^2 + \cancel{2\beta x^0 x^1} - \beta^2(x^0)^2\Big] \\ &= \gamma^2\left[(x^0)^2(1-\beta^2) - (x^1)^2(1-\beta^2)\right] \\ &= \gamma^2(1-\beta^2)\left[(x^0)^2 - (x^1)^2\right] = (x^0)^2 - (x^1)^2.\end{aligned}$$

Since $x^{2\prime} = x^2$ and $x^{3\prime} = x^3$, it follows that

$$(x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \quad \checkmark$$

Problem 3.9

$\boxed{a_\mu = (3, -4, -1, -2)}$
$\boxed{b_\mu = (5, 0, -3, -4)}$
$\mathbf{a}^2 = 4^2 + 1^2 + 2^2 = \boxed{21}$
$\mathbf{b}^2 = 0^2 + 3^2 + 4^2 = \boxed{25}$
$\mathbf{a} \cdot \mathbf{b} = (4)(0) + (1)(3) + (2)(4) = \boxed{11}$
$a^2 = 3^2 - \mathbf{a}^2 = 9 - 21 = \boxed{\text{-12,}}$ so a^μ is $\boxed{\text{spacelike.}}$
$b^2 = 5^2 - \mathbf{b}^2 = 25 - 25 = \boxed{0,}$ so b^μ is $\boxed{\text{lightlike.}}$
$a \cdot b = (3)(5) - \mathbf{a} \cdot \mathbf{b} = 15 - 11 = \boxed{4.}$

Problem 3.10

(a) $\boxed{10}$ $(s^{00}, s^{11}, s^{22}, s^{33}$, and $s^{01}, s^{02}, s^{03}, s^{12}, s^{13}, s^{23})$

(b) $\boxed{6}$ (the latter six – the first four are zero in this case).

(c)

$$\begin{aligned} s^{\nu\mu\prime} &= \Lambda^\nu_\kappa \Lambda^\mu_\sigma s^{\kappa\sigma} = \Lambda^\nu_\kappa \Lambda^\mu_\sigma s^{\sigma\kappa} \\ &= \Lambda^\mu_\sigma \Lambda^\nu_\kappa s^{\sigma\kappa} = s^{\mu\nu\prime}, \quad \text{so } s^{\mu\nu\prime} \text{ is symmetric.} \end{aligned}$$

$$\begin{aligned} a^{\nu\mu\prime} &= \Lambda^\nu_\kappa \Lambda^\mu_\sigma a^{\kappa\sigma} = \Lambda^\nu_\kappa \Lambda^\mu_\sigma (-a^{\sigma\kappa}) \\ &= -\Lambda^\mu_\sigma \Lambda^\nu_\kappa a^{\sigma\kappa} = -a^{\mu\nu\prime}, \end{aligned}$$

so antisymmetry is also preserved by Lorentz transformations.

(d)

$$\begin{aligned} s_{\nu\mu} &= g_{\nu\kappa} g_{\mu\lambda} s^{\kappa\lambda} = g_{\nu\kappa} g_{\mu\lambda} s^{\lambda\kappa} \\ &= g_{\mu\lambda} g_{\nu\kappa} s^{\lambda\kappa} = s_{\mu\nu}, \quad \text{so } s_{\mu\nu} \text{ is symmetric.} \end{aligned}$$

$$\begin{aligned} a_{\nu\mu} &= g_{\nu\kappa} g_{\mu\lambda} a^{\kappa\lambda} = g_{\nu\kappa} g_{\mu\lambda} (-a^{\lambda\kappa}) \\ &= -g_{\mu\lambda} g_{\nu\kappa} a^{\lambda\kappa} = -a_{\mu\nu}, \quad \text{so } a_{\mu\nu} \text{ is antisymmetric.} \end{aligned}$$

(e)

$$\begin{aligned} s^{\mu\nu}a_{\mu\nu} &= s^{\nu\mu}a_{\nu\mu} \quad \text{(just by renaming the summation indices)} \\ &= s^{\mu\nu}(-a_{\mu\nu}) \quad \text{(by symmetry/antisymmetry)} \\ &= -(s^{\mu\nu}a_{\mu\nu}). \quad \text{But if } x = -x, \text{ then } x = 0. \end{aligned}$$

(f) Let $\boxed{s^{\mu\nu} = \frac{1}{2}(t^{\mu\nu} + t^{\nu\mu})}$ and $\boxed{a^{\mu\nu} = \frac{1}{2}(t^{\mu\nu} - t^{\nu\mu}).}$
Clearly $s^{\mu\nu} = s^{\nu\mu}$, $a^{\mu\nu} = -a^{\nu\mu}$, and $t^{\mu\nu} = s^{\mu\nu} + a^{\mu\nu}$.

Problem 3.11

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{3}{5}\right)^2}} = \frac{1}{\sqrt{\frac{16}{25}}} = \frac{5}{4}.$$

$$\eta^\mu = \gamma(c, v_x, v_y, v_z) = \frac{5}{4}\left(c, \frac{3}{5}c, 0, 0\right) = \boxed{\frac{c}{4}(5, 3, 0, 0)}$$

Problem 3.12

$$p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$$

Multiply by Λ^ν_μ to transform to S':

$$\begin{aligned} \Lambda^\nu_\mu(p_A^\mu + p_B^\mu) &= \Lambda^\nu_\mu(p_C^\mu + p_D^\mu) \\ p_A^{\nu\,\prime} + p_B^{\nu\,\prime} &= p_C^{\nu\,\prime} + p_D^{\nu\,\prime} \end{aligned}$$

Problem 3.13

$p^2 = m^2c^2 > 0$ (Eq. 3.43), so it's $\boxed{\text{timelike}}$ (Eq. 3.25); if $m = 0$, then $p^2 = 0$, so it's $\boxed{\text{lightlike}}$. The 4-momentum of a *virtual* particle could be $\boxed{\text{anything}}$.

Problem 3.14

Say the potato weighs $\frac{1}{4}$ kg at room temperature (20°C), and we heat it to the boiling point (100°C). If it were pure water (specific heat $c = 1$ Cal/kg°C=

4000 J/kg°C), the heat required would be

$$Q = mc\Delta T = (4000\,\text{J/kg}^\circ\text{C})\left(\frac{1}{4}\,\text{kg}\right)(80^\circ\text{C}) = 8\times 10^4\,\text{J}.$$

The increase in mass is

$$\Delta m = \frac{Q}{c^2} = \frac{8\times 10^4\,\text{J}}{(3\times 10^8\,\text{m/s})^2} \cong \boxed{10^{-12}\,\text{kg}}. \text{ (Not a substantial addition!)}$$

Problem 3.15

$$p_\pi = p_\mu + p_\nu \quad \text{(4-vectors)} \Longrightarrow p_\mu = p_\pi - p_\nu \Longrightarrow p_\mu^2 = p_\pi^2 + p_\nu^2 - 2p_\pi\cdot p_\nu.$$

$$p_\mu^2 = m_\mu^2 c^2, \quad p_\pi^2 = m_\pi^2 c^2, \quad p_\nu^2 = 0;$$

$$p_\pi \cdot p_\nu = \frac{E_\pi}{c}\frac{E_\nu}{c} - \mathbf{p}_\pi\cdot\mathbf{p}_\nu, \quad \text{but} \quad \mathbf{p}_\pi\cdot\mathbf{p}_\nu = 0, \quad \text{so} \quad m_\mu^2 c^2 = m_\pi^2 c^2 - 2\frac{E_\pi E_\nu}{c^2}.$$

But $E_\pi = \gamma m_\pi c^2$ and $E_\nu = |\mathbf{p}_\nu|c$. So

$$(m_\pi^2 - m_\mu^2)c^2 = 2\gamma m_\pi |\mathbf{p}_\nu| c \Longrightarrow |\mathbf{p}_\nu| = \frac{(m_\pi^2 - m_\mu^2)c}{2\gamma m_\pi}. \quad \text{Also,} \quad |\mathbf{p}_\pi| = \gamma m_\pi v.$$

$$\text{So} \quad \tan\theta = \frac{|\mathbf{p}_\nu|}{|\mathbf{p}_\pi|} = \frac{(m_\pi^2 - m_\mu^2)c}{2\gamma m_\pi \gamma m_\pi v} = \boxed{\frac{\left[1 - (m_\mu^2/m_\pi^2)\right]}{2\beta\gamma^2}} \quad \left(\text{where } \beta = \frac{v}{c}\right).$$

Problem 3.16

Before the collision, in the lab frame, $p^\mu_{\text{TOT}} = \left(\frac{E_A}{c} + m_B c,\ \mathbf{p}_A\right)$, so

$$p_{\text{TOT}}{}^2 = \left(\frac{E_A}{c} + m_B c\right)^2 - \mathbf{p}_A^2 = \frac{E_A^2}{c^2} + 2E_A m_B + m_B^2 c^2 - \mathbf{p}_A^2$$
$$= 2E_A m_B + (m_B^2 + m_A^2)c^2$$

(I used $E_A^2 - \mathbf{p}_A^2 c^2 = m_A^2 c^4$). After the collision, in the *CM* frame, at threshold:

$$p^\mu_{\text{TOT}}{}' = ((m_1 + m_2 + \ldots + m_n)c,\ \mathbf{0}) = (Mc,\ \mathbf{0}); \quad (p_{\text{TOT}}{}')^2 = (Mc)^2.$$

But p^μ_{TOT} is conserved (same before as after, in either frame) and $p_{\text{TOT}}{}^2$ is invariant (same value in both frames), so

$$2E_A m_B + (m_A^2 + m_B^2)c^2 = M^2 c^2, \quad \text{or} \quad \boxed{E_A = \frac{(M^2 - m_A^2 - m_B^2)c^2}{2m_B}.}$$

[Note that this generalizes the result in Example 3.4. There, $m_A = m_B = m_p$; $M = 4m_p$; so $E_A = \frac{(16-1-1)m_p^2c^2}{2m_p} = 7m_pc^2$.]

Problem 3.17

$$E_A = \frac{(M^2 - m_A^2 - m_B^2)c^2}{2m_B}$$

(a)

$$M = 2m_p + m_{\pi^0} = 2(938.27) + 134.98 = 2011.52$$

$$m_A = m_B = m_p = 938.27$$

$$E = \frac{(2011.52)^2 - 2(938.27)^2}{2(938.27)} = \boxed{1218 \text{ MeV}}$$

(b)

$$M = 2m_p + m_{\pi^+} + m_{\pi^-} = 2(938.27) + 2(139.57) = 2155.68$$

$$m_A = m_B = m_p = 938.27$$

$$E = \frac{(2155.68)^2 - 2(938.27)^2}{2(938.27)} = \boxed{1538 \text{ MeV}}$$

(c)

$$M = 2m_p + m_n = 2(938.27) + 939.57 = 2816.11$$

$$m_A = m_{\pi^-} = 139.57; \quad m_B = m_p = 938.27$$

$$E = \frac{(2816.11)^2 - (139.57)^2 - (938.27)^2}{2(938.27)} = \boxed{3747 \text{ MeV}}$$

(d)

$$M = m_{K^0} + m_{\Sigma^0} = 497.65 + 1192.6 = 1690.3$$

$$m_A = m_{\pi^-} = 139.57; \quad m_B = m_p = 938.27$$

$$E = \frac{(1690.3)^2 - (139.57)^2 - (938.27)^2}{2(938.27)} = \boxed{1043 \text{ MeV}}$$

(e)

$$M = m_p + m_{\Sigma^+} + m_{K^0} = 938.27 + 1189.4 + 497.65 = 2625.3$$

$$m_A = m_B = m_p = 938.27$$

$$E = \frac{(2625.3)^2 - 2(938.27)^2}{2(938.27)} = \boxed{2735\ \text{MeV}}$$

Problem 3.18

For the second reaction ($K^- + p \to \Omega^- + K^0 + K^+$),

$$M = m_{\Omega^-} + m_{K^0} + m_{K^+} = 1672.45 + 497.67 + 493.68 = 2663.80,$$

$$m_A = m_{K^-} = 493.68 \quad \text{and} \quad m_B = m_p = 938.27,$$

so the threshold energy for the K^- is

$$E_{K^-} = \frac{(2663.80)^2 - (493.68)^2 - (938.27)^2}{2(938.27)} = 3182.3\ \text{MeV}.$$

In the first reaction, at threshold, the K^- goes in the forward direction, and the other particles emerge as a group (there is no point in "wasting" energy in transverse motion, or in internal motion of the p, p, K^+).

We have, in effect, an outgoing "particle" C, of mass

$$m_C = 2m_p + m_K = 2(938.27) + 493.68 = 2370.22\,\text{MeV}/c^2.$$

Conservation of energy/momentum says $p_1 + p_2 = p_C + p_K$, or $p_1 - p_K = p_C - p_2$. Squaring:

$$p_1^2 + p_K^2 - 2p_1 \cdot p_K = p_C^2 + p_2^2 - 2p_C \cdot p_2,$$

$$\cancel{m_p^2 c^2} + m_K^2 c^2 - 2\left(\frac{E}{c}\frac{E_K}{c} - \mathbf{p}_1 \cdot \mathbf{p}_K\right) = m_C^2 c^2 + \cancel{m_p^2 c^2} - 2\left(\frac{E_C}{c} m_p c\right).$$

Now

$$E_C = E + m_p c^2 - E_K, \quad \mathbf{p}_1 \cdot \mathbf{p}_K = |\mathbf{p}_1||\mathbf{p}_K|, \quad |\mathbf{p}_1| = \frac{1}{c}\sqrt{E^2 - (m_p c^2)^2},$$

and

$$|\mathbf{p}_K| = \frac{1}{c}\sqrt{E_K^2 - (m_K c^2)^2} = \frac{1}{c}\sqrt{(3182.3)^2 - (439.68)^2} = \frac{1}{c}(3143.8) \equiv \frac{a}{c}.$$

$$-EE_K + a\sqrt{E^2 - (m_p c^2)^2} = \tfrac{1}{2}\left[(m_C c^2)^2 - (m_K c^2)^2\right] - (m_p c^2)(E + m_p c^2 - E_K),$$

$$a\sqrt{E^2 - (m_p c^2)^2} = E(E_K - m_p c^2) + b,$$

where

$$b \equiv \tfrac{1}{2}\left[(m_C c^2)^2 - (m_K c^2)^2\right] + (m_p c^2)(E_K - m_p c^2) = 4.7926 \times 10^6.$$

Squaring,

$$a^2\left[E^2 - (m_p c^2)^2\right] = E^2\left(E_K - m_p c^2\right)^2 + b^2 + 2bE\left(E_K - m_p c^2\right),$$

$$E^2\left[a^2 - \left(E_K - m_p c^2\right)^2\right] - 2bE\left(E_K - m_p c^2\right) - \left[b^2 + a^2\left(m_p c^2\right)^2\right] = 0.$$

Numerically,

$$\begin{aligned} E_K - m_p c^2 &= 2244.0, \\ a^2 - \left(E_K - m_p c^2\right)^2 &= 4.84775 \times 10^6, \\ 2b(E_K - m_p c^2) &= 2.15091 \times 10^{10}, \\ b^2 + a^2\left(m_p c^2\right)^2 &= 3.16697 \times 10^{13}. \end{aligned}$$

So

$$E^2\left(4.84775 \times 10^6\right) - E\left(2.15091 \times 10^{10}\right) - \left(3.16697 \times 10^{13}\right) = 0,$$

or

$$E^2 - E(4436.9) - \left(6.5329 \times 10^6\right) = 0.$$

$$E = \frac{1}{2}\left[4436.9 \pm \sqrt{(4436.9)^2 + 4(6.5329 \times 10^6)}\right] = \frac{1}{2}(4436.9 \pm 6768.9).$$

But E has to be positive, so we need the upper sign:

$$E = 5602.9 \Rightarrow T = 5602.9 - 938.3 = \boxed{4664.6\,\text{MeV}}.$$

Problem 3.19

(a) $p^\mu_A = p^\mu_B + p^\mu_C$, or $p^\mu_C = p^\mu_A - p^\mu_B$. Square both sides: $p_C^2 = p_A^2 + p_B^2 - 2p_A \cdot p_B$.

$$\text{Now,}\quad p_A^2 = m_A^2 c^2, \quad p_B^2 = m_B^2 c^2, \quad p_C^2 = m_C^2 c^2, \quad p_A \cdot p_B = \frac{E_A}{c}\frac{E_B}{c} - \mathbf{p}_A \cdot \mathbf{p}_B.$$

$$\text{But}\quad \mathbf{p}_A = 0, \text{ and } E_A = m_A c^2, \text{so } m_C^2 c^2 = m_A^2 c^2 + m_B^2 c^2 - 2m_A E_B, \quad \text{so}$$

$$2m_A E_B = -(m_C^2 - m_A^2 - m_B^2)c^2, \quad \text{or}$$

$$\boxed{E_B = \frac{(-m_C^2 + m_A^2 + m_B^2)}{2m_A}c^2}; \quad \boxed{E_C = \frac{(m_A^2 - m_B^2 + m_C^2)}{2m_A}c^2}$$

(b)

$$E_B^2 - \mathbf{p}_B^2 c^2 = m_B^2 c^4 \Longrightarrow \mathbf{p}_B^2 = \frac{E_B^2}{c^2} - m_B^2 c^2 = \frac{(m_A^2 + m_B^2 - m_C^2)^2 c^2}{4m_A^2} - \frac{4m_A^2 m_B^2 c^2}{4m_A^2}$$

$$\begin{aligned} |\mathbf{p}_B| &= \frac{c}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 + 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2 - 4m_A^2 m_B^2} \\ &= \frac{c}{2m_A}\sqrt{\lambda(m_A^2,\, m_B^2,\, m_C^2)} \end{aligned}$$

$$\therefore \boxed{|\mathbf{p}_B| = |\mathbf{p}_C| = \frac{c}{2m_A}\sqrt{\lambda(m_A^2,\, m_B^2,\, m_C^2)}}$$

(c) The decay is kinematically forbidden if $m_A < m_B + m_C$ (not enough energy to produce the final particles, in *CM* frame).

Problem 3.20

(a) $\boxed{E_\mu = 109.8 \text{ MeV}; \quad E_\nu = 29.8 \text{ MeV}}$

(b) $\boxed{E_\gamma = 67.5 \text{ MeV}}$

(c) $\boxed{E_{\pi^+} = 248.1 \text{ MeV}; \quad E_{\pi^0} = 245.6 \text{ MeV}}$

(d) $\boxed{E_p = 943.6 \text{ MeV}; \quad E_{\pi^-} = 172.1 \text{ MeV}}$

(e) $\boxed{E_\Lambda = 1135 \text{ MeV}; \quad E_{K^+} = 536.6 \text{ MeV}}$

Problem 3.21

The velocity of the muon is $v = \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right) c$ (Example 3.3). So

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right)^2}} = \frac{(m_\pi^2 + m_\mu^2)}{\sqrt{(m_\pi^2 + m_\mu^2)^2 - (m_\pi^2 - m_\mu^2)^2}}$$

$$= \frac{(m_\pi^2 + m_\mu^2)}{\sqrt{m_\pi^4 + m_\mu^4 + 2m_\pi^2 m_\mu^2 - m_\pi^4 - m_\mu^4 + 2m_\pi^2 m_\mu^2}} = \frac{m_\pi^2 + m_\mu^2}{2m_\pi m_\mu}.$$

Lab lifetime is $\gamma\tau$, so distance is $v\gamma\tau$, or

$$d = \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right) c \frac{m_\pi^2 + m_\mu^2}{2m_\pi m_\mu} \tau = \frac{m_\pi^2 - m_\mu^2}{2m_\pi m_\mu} c\tau.$$

$$d = \frac{(139.6)^2 - (105.7)^2}{2(139.6)(105.7)} (3 \times 10^8 \text{ m/s})(2.20 \times 10^{-6} \text{ s}) = \boxed{186 \text{ m.}}$$

Problem 3.22

(a) The *minimum* is clearly $\boxed{E_{\text{min}} = m_B c^2}$, with B at rest. For *maximum* E_B we want B going one way, C, D, ... moving as a unit in the opposite direction (there is no point in wasting energy in relative motion of the other particles). Thus the others act as a *single* particle, of mass $M = m_C + m_D + \cdots$, and we can simply quote the result for *two*-body decays (Problem 3.19a):

$$\boxed{E_{\text{max}} = \frac{m_A^2 + m_B^2 - M^2}{2m_A} c^2.}$$

In case that argument is not completely compelling, let's look more closely at the case of just *three* outgoing particles. For maximum E_B we certainly want C and D to come off *opposite* to B, and the only question is how best to *apportion* the momentum between them. Let's say

$$\mathbf{p}_C = -x\mathbf{p}_B, \quad \mathbf{p}_D = (x-1)\mathbf{p}_B \quad (\text{so that } \mathbf{p}_C + \mathbf{p}_D = -\mathbf{p}_B).$$

Here x ranges from 0 (C at rest) to 1 (D at rest). From conservation of energy, $m_A c^2 = E_B + E_C + E_D$. But

$$E_C = \sqrt{m_C^2 c^4 + x^2 \mathbf{p}_B^2 c^2}, \quad E_D = \sqrt{m_D^2 c^4 + (x-1)^2 \mathbf{p}_B^2 c^2},$$

and $\mathbf{p}_B^2 c^2 = E_B^2 - m_B^2 c^4$. So

$$m_A c^2 = E_B + \sqrt{m_C^2 c^4 + x^2 (E_B^2 - m_B^2 c^4)} + \sqrt{m_D^2 c^4 + (x-1)^2 (E_B^2 - m_B^2 c^4)}.$$

Differentiate with respect to x:

$$0 = \frac{dE_B}{dx} + \frac{1}{2}\frac{1}{\sqrt{\bigstar}}\left[2x(E_B^2 - m_B^2 c^4) + x^2 2E_B \frac{dE_B}{dx}\right] \\ + \frac{1}{2}\frac{1}{\sqrt{\blacklozenge}}\left[2(x-1)(E_B^2 - m_B^2 c^4) + (x-1)^2 2E_B \frac{dE_B}{dx}\right],$$

$$\sqrt{\bigstar} \equiv \sqrt{m_C^2 c^4 + x^2 (E_B^2 - m_B^2 c^4)}; \sqrt{\blacklozenge} \equiv \sqrt{m_D^2 c^4 + (x-1)^2 (E_B^2 - m_B^2 c^4)}.$$

For the maximum we want $dE_B/dx = 0$, so

$$0 = \frac{1}{2\sqrt{\bigstar}} 2x(E_B^2 - m_B^2 c^4) + \frac{1}{2\sqrt{\blacklozenge}} 2(x-1)(E_B^2 - m_B^2 c^4),$$

$$\frac{x}{\sqrt{\bigstar}} + \frac{(x-1)}{\sqrt{\blacklozenge}} = 0, \quad x\sqrt{\blacklozenge} = (1-x)\sqrt{\bigstar}$$

Therefore

$$x^2\left[m_D^2 c^4 + (x-1)^2 (E_B^2 - m_B^2 c^4)\right] = (1-x)^2\left[m_C^2 c^4 + x^2 (E_B^2 - m_B^2 c^4)\right],$$

$$x^2 m_D^2 c^4 = (1-x)^2 m_C^2 c^4 \implies x m_D = (1-x) m_C; \quad x(m_C + m_D) = m_C,$$

$$x = \frac{m_C}{(m_C + m_D)}.$$

What is the corresponding E_B?

$$m_A c^2 = E_B + \sqrt{\bigstar} + \sqrt{\blacklozenge}, \quad \text{but} \quad x\sqrt{\blacklozenge} = (1-x)\sqrt{\bigstar}, \quad \text{so}$$

$$\sqrt{\blacklozenge} = \left(\frac{1-x}{x}\right)\sqrt{\bigstar} = \left(\frac{1}{x} - 1\right)\sqrt{\bigstar}, \quad \text{and hence}$$

$$m_A c^2 = E + \sqrt{\bigstar} + \left(\frac{1}{x} - 1\right)\sqrt{\bigstar} = E + \frac{1}{x}\sqrt{\bigstar} \\ = E + \sqrt{\frac{1}{x^2} m_C^2 c^4 + E^2 - m_B^2 c^4}. \quad (E \equiv E_B)$$

$$(m_A c^2 - E)^2 = (m_C + m_D)^2 c^4 + E^2 - m_B^2 c^4$$
$$\Longrightarrow m_A^2 c^4 - 2E m_A c^2 + \cancel{E^2} = (m_C + m_D)^2 c^4 + \cancel{E^2} - m_B^2 c^4$$
$$2E m_A = \left[m_A^2 + m_B^2 - (m_C + m_D)^2\right] c^2.$$
$$\boxed{E_{\max} = \frac{\left[m_A^2 + m_B^2 - (m_C + m_D)^2\right]}{2m_A} c^2.}$$

Are we sure this is a *maximum*? Maybe it's a *minimum*, and the maximum occurs at the end of the interval ($x = 0$ or $x = 1$). We can test for this by calculating E at the end points: At $x = 0$,

$$m_A c^2 = E + m_C c^2 + \sqrt{m_D^2 c^4 + E^2 - m_B^2 c^4}.$$
$$(m_A - m_C)c^2 - E = \sqrt{(m_D^2 - m_B^2)c^4 + E^2}$$
$$(m_A - m_C)^2 c^4 - 2E(m_A - m_C)c^2 + \cancel{E^2} = (m_D^2 - m_B^2)c^4 + \cancel{E^2}$$
$$E_0 = \frac{(m_A - m_C)^2 + m_B^2 - m_D^2}{2(m_A - m_C)} c^2$$

(For $x = 1$, just interchange C and D.) Now

$$\frac{2}{c^2}(E_{\max} - E_0) = \cancel{m_A} + \frac{m_B^2}{m_A} - \frac{(m_C + m_D)^2}{m_A} - \cancel{m_A} + m_C - \frac{(m_B^2 - m_D^2)}{(m_A - m_C)}$$
$$= \frac{(m_A - m_C)\left[m_B^2 - (m_C + m_D)^2\right] + m_A m_C (m_A - m_C) - m_A (m_B^2 - m_D^2)}{m_A (m_A - m_C)}.$$

$$\text{So} \quad \frac{2m_A(m_A - m_C)}{c^2}(E_{\max} - E_0)$$
$$= m_A\left[\cancel{m_B^2} - m_C^2 - \cancel{m_D^2} - 2m_C m_D + m_A m_C - m_C^2 - \cancel{m_B^2} + \cancel{m_D^2}\right]$$
$$- m_C(m_B^2 - m_C^2 - m_D^2 - 2m_C m_D)$$
$$= m_C\left[m_A(-2m_C - 2m_D + m_A) - m_B^2 + m_C^2 + m_D^2 + 2m_C m_D\right].$$

$$\frac{2m_A(m_A - m_C)}{m_C c^2}(E_{\max} - E_0)$$
$$= (m_A^2 + m_C^2 + m_D^2 - 2m_A m_C - 2m_A m_D + 2m_C m_D) - m_B^2$$
$$= (m_A - m_C - m_D)^2 - m_B^2.$$

But $m_A > m_B + m_C + m_D \Longrightarrow (m_A - m_C - m_D) > m_B$,

so the right side is *positive*, and therefore $E_{\max} > E_0$. (By the same token, $E_{\max} > E_1$ (at $x = 1$), which differs only by $C \leftrightarrow D$.) Conclusion: $E_{\max}$ is a *maximum*, not a minimum.

(b) $\boxed{E_{\text{min}} = m_e c^2 = 0.511 \text{ MeV}}$; $\boxed{E_{\text{max}} = \frac{(m_\mu^2 + m_e^2)}{2m_\mu} c^2 = 52.8\text{MeV}.}$

Problem 3.23

(a) The CM moves at speed v relative to lab. *Classically*, $u = v + v = 2v$, but *relativistically* the velocities add by Eq. 3.5:

$$u = \frac{v+v}{1+\frac{vv}{c^2}} = \frac{2v}{1+\frac{v^2}{c^2}}. \quad \text{Solve for } v:$$

$$u + u\frac{v^2}{c^2} = 2v \Longrightarrow \frac{u}{c^2}v^2 - 2v + u = 0$$

$$\Longrightarrow v = \frac{+2 \pm \sqrt{4 - 4\frac{u^2}{c^2}}}{2\frac{u}{c^2}} = \frac{c^2}{u}\left(1 \pm \sqrt{1 - \frac{u^2}{c^2}}\right)$$

If we use the $+$ sign, for small u we get $v \approx 2\frac{c^2}{u}$, which is wrong (it should be $\frac{u}{2}$), so we need the $-$ sign:

$$\boxed{v = \frac{c^2}{u}\left(1 - \sqrt{1 - \frac{u^2}{c^2}}\right).}$$

(b) In the CM frame, then:

$$\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}} = \frac{1}{1 - \frac{c^2}{u^2}\left(1 - 2\sqrt{1-\frac{u^2}{c^2}} + 1 - \frac{u^2}{c^2}\right)}$$

$$= \frac{1}{1 - 2\frac{c^2}{u^2} + 1 + 2\frac{c^2}{u^2}\sqrt{1-\frac{u^2}{c^2}}} = \frac{\frac{u^2}{c^2}}{2\left[\sqrt{1-\frac{u^2}{c^2}} - \left(1 - \frac{u^2}{c^2}\right)\right]}.$$

Let $\gamma' \equiv \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ (the γ in the lab frame). Then

$$\sqrt{1-\frac{u^2}{c^2}} = \frac{1}{\gamma'}; \qquad \frac{u^2}{c^2} = 1 - \frac{1}{(\gamma')^2}. \quad \text{Therefore}$$

$$\gamma^2 = \frac{\left[1 - \frac{1}{(\gamma')^2}\right]}{2\left[\frac{1}{\gamma'} - \frac{1}{(\gamma')^2}\right]} = \frac{(\gamma'^2 - 1)}{2(\gamma' - 1)} = \frac{(\gamma'-1)(\gamma'+1)}{2(\gamma'-1)} = \frac{1}{2}(\gamma'+1)$$

$$\boxed{\gamma = \sqrt{\tfrac{1}{2}(\gamma'+1)}}$$

(c)

So $\quad T = (\gamma - 1)mc^2 = \boxed{\left[\frac{1}{\sqrt{2}}\sqrt{\gamma' + 1} - 1\right] mc^2.}$

Hence $\quad \frac{1}{2}(\gamma' + 1) = \left(\frac{T}{mc^2} + 1\right)^2.\quad$ Therefore

$$\gamma' = -1 + 2\left(\frac{T^2}{m^2c^4} + \frac{2T}{mc^2} + 1\right) = \frac{2T^2}{m^2c^4} + \frac{4T}{mc^2} + 1.$$

$T' = (\gamma' - 1)mc^2 \quad$ (kinetic energy in lab), $\quad$ so

$$T' = \left(\frac{2T^2}{m^2c^4} + \frac{4T}{mc^2}\right) mc^2 = \frac{2T^2}{mc^2} + 4T = 4T\left(1 + \frac{T}{2mc^2}\right). \quad \checkmark$$

Problem 3.24

Let p_A be the 4-momentum of A *before* the collision, and let q_A be the 4-momentum *after* the collision, in the CM frame; p'_A, q'_A are the corresponding quantities in the Breit fame. Now $p_A \cdot q_A = p'_A \cdot q'_A$ (since the 4-dimensional dot product is invariant):

$$\frac{E_A^2}{c^2} - \mathbf{p}_A \cdot \mathbf{q}_A = \frac{E_A'^2}{c^2} - \mathbf{p}'_A \cdot \mathbf{q}'_A.$$

(*Note*: The incoming A and outgoing A have the same energy in CM – only the *direction* of A's momentum changes. Likewise, the incoming and outgoing A have same energy (E'_A) in Breit frame, since their momenta are opposite.) Now $\mathbf{p}_A \cdot \mathbf{q}_A = \mathbf{p}_A^2 \cos\theta$, where

$$\mathbf{p}_A^2 = \frac{1}{c^2}\left(E_A^2 - m_A^2c^4\right), \text{ and } \mathbf{p}'_A \cdot \mathbf{q}'_A = -\mathbf{p}_A'^2 = -\frac{1}{c^2}\left(E_A'^2 - m_A^2c^4\right).$$

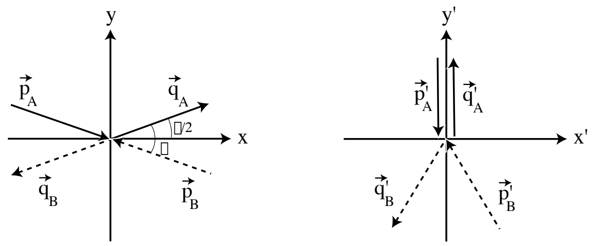
So

$$\frac{E_A^2}{c^2} - \frac{1}{c^2}\left(E_A^2 - m_A^2c^4\right)\cos\theta = \frac{E_A'^2}{c^2} + \frac{1}{c^2}\left(E_A'^2 - m_A^2c^4\right);$$

$$E_A^2(1 - \cos\theta) + m_A^2c^4\cos\theta = 2E_A'^2 - m_A^2c^4.$$

$$E_A'^2 = E_A^2\left(\frac{1 - \cos\theta}{2}\right) + m_A^2c^4\left(\frac{1 + \cos\theta}{2}\right) = E_A^2\sin^2\frac{\theta}{2} + m_A^2c^4\cos^2\frac{\theta}{2}$$

or (dropping the subscripts) $\boxed{E' = \sqrt{E^2\sin^2\frac{\theta}{2} + m^2c^4\cos^2\frac{\theta}{2}}.}$



$\boxed{\text{Velocity makes angle } \theta/2 \text{ with incident } A.}$ (**v** points along the line bisecting $\mathbf{p}_A$ and $\mathbf{q}_A$ – see diagram.)

$$q'_{A_x} = \gamma\left(q_{A_x} - \frac{v}{c}\frac{E}{c}\right) = 0$$

$$\Longrightarrow v = \frac{c^2 q_{A_x}}{E} = \frac{c^2|\mathbf{q}_A|\cos\frac{\theta}{2}}{E} = \frac{c^2}{E}\cos\frac{\theta}{2}\left(\frac{1}{c}\sqrt{E^2 - m_A^2 c^4}\right)$$

$$\boxed{v = \left(\cos\frac{\theta}{2}\right) c\sqrt{1 - \left(\frac{m_A c^2}{E}\right)^2}}$$

Problem 3.25

(a)

$$\begin{aligned} s + t + u &= \frac{1}{c^2}\left[(p_A + p_B)^2 + (p_A - p_C)^2 + (p_A - p_D)^2\right] \\ &= \frac{1}{c^2}\left[p_A^2 + 2p_A \cdot p_B + p_B^2 + p_A^2 - 2p_A \cdot p_C + p_C^2 + p_A^2 - 2p_A \cdot p_D + p_D^2\right] \\ &= \frac{1}{c^2}\left[m_A^2c^2 + m_B^2c^2 + m_C^2c^2 + m_D^2c^2 + 2p_A \cdot (p_A + p_B - p_C - p_D)\right] \end{aligned}$$

Conservation of energy and momentum requires $p_A + p_B = p_C + p_D$, so

$$\boxed{s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2.} \quad \checkmark$$

(b)

$$c^2 s = p_A^2 + 2p_A \cdot p_B + p_B^2 = p_A^2 + 2\left(\frac{E_A}{c}\frac{E_B}{c} - \mathbf{p}_A \cdot \mathbf{p}_B\right) + p_B^2$$

In the CM frame, $\mathbf{p}_A \cdot \mathbf{p}_B = -\mathbf{p}_A^2$, so

$$p_A^2 = \frac{E_A^2}{c^2} - \mathbf{p}_A^2 \implies \mathbf{p}_A^2 = \frac{E_A^2}{c^2} - p_A^2 = -\mathbf{p}_A \cdot \mathbf{p}_B$$

$$\text{and} \quad p_B^2 = \frac{E_B^2}{c^2} - \mathbf{p}_B^2 = \frac{E_B^2}{c^2} - \mathbf{p}_A^2$$

$$\implies \frac{E_B}{c} = \sqrt{p_B^2 + \mathbf{p}_A^2} = \sqrt{p_B^2 + \frac{E_A^2}{c^2} - p_A^2}.$$

So

$$c^2 s = p_A^2 + 2\left(\frac{E_A}{c}\sqrt{p_B^2 + \frac{E_A^2}{c^2} - p_A^2} + \frac{E_A^2}{c^2} - p_A^2\right) + p_B^2$$

$$c^2 s + p_A^2 - p_B^2 - 2\frac{E_A^2}{c^2} = 2\frac{E_A}{c}\sqrt{p_B^2 + \frac{E_A^2}{c^2} - p_A^2}$$

Squaring both sides,

$$(c^2 s + p_A^2 - p_B^2)^2 - \frac{4}{c^2}(c^2 s + p_A^2 - p_B^2)E_A^2 + \cancel{4\frac{E_A^4}{c^4}} = \frac{4}{c^2}(p_B^2 - p_A^2)E_A^2 + \cancel{4\frac{E_A^4}{c^4}}$$

$$E_A^2 = \frac{(c^2 s + p_A^2 - p_B^2)^2}{4s}$$

$$\boxed{E_A^{\text{CM}} = \frac{(s + m_A^2 - m_B^2)c^2}{2\sqrt{s}}}.$$

(c) In the lab frame, $\mathbf{p}_B = 0$ and $E_B = m_B c^2$, so

$$c^2 s = p_A^2 + 2\frac{E_A}{c}\frac{m_B c^2}{c} + p_B^2 = m_A^2 c^2 + 2m_B E_A + m_B c^2$$

$$\boxed{E_A^{\text{lab}} = \frac{(s - m_A^2 - m_B^2)c^2}{2m_B}}.$$

(d) In the CM frame, $(p_A + p_B)^2 = \left(\frac{E_A}{c} + \frac{E_B}{c}\right)^2 = \frac{E_{\text{TOT}}^2}{c^2} \implies \boxed{E_{\text{TOT}}^{\text{CM}} = \sqrt{s}c^2}$

Problem 3.26

$$s = \frac{1}{c^2}(p_A + p_B)^2 = \frac{1}{c^2}\left[\left(\frac{E_A + E_B}{c}\right)^2 - (\mathbf{p}_A + \mathbf{p}_B)^2\right]$$

In the CM frame, $\mathbf{p}_A + \mathbf{p}_B = 0$ and $E_A = E_B = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$, so

$$s = \frac{1}{c^2}\left(\frac{4(\mathbf{p}^2c^2 + m^2c^4)}{c^2}\right) = \boxed{\frac{4(\mathbf{p}^2 + m^2c^2)}{c^2}.} \quad \checkmark$$

$$t = \frac{1}{c^2}(p_A - p_C)^2 = \frac{1}{c^2}\left[\left(\frac{E_A - E_C}{c}\right)^2 - (\mathbf{p}_A - \mathbf{p}_C)^2\right]$$

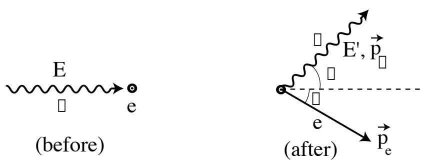
But $E_A = E_C$ and $(\mathbf{p}_A - \mathbf{p}_C)^2 = \mathbf{p}_A^2 + \mathbf{p}_C^2 - 2\mathbf{p}_A \cdot \mathbf{p}_C = 2\mathbf{p}^2(1 - \cos\theta)$, so

$$\boxed{t = \frac{-2\mathbf{p}^2(1 - \cos\theta)}{c^2}.} \quad \checkmark$$

u is the same as t, except that $\mathbf{p}_A \cdot \mathbf{p}_D = -\mathbf{p}_A \cdot \mathbf{p}_D = -\mathbf{p}^2\cos\theta$:

$$\boxed{u = \frac{-2\mathbf{p}^2(1 + \cos\theta)}{c^2}.} \quad \checkmark$$

Problem 3.27



In this problem I'll use p_e for the *three*-momentum of the charged particle, and p_γ for the *three*-momentum of the photon.

Conservation of momentum:

$$p_e \sin\phi = p_\gamma \sin\theta \Longrightarrow \sin\phi = \frac{E'}{cp_e}\sin\theta$$

$$\frac{E}{c} = p_\gamma\cos\theta + p_e\cos\phi = \frac{E'}{c}\cos\theta + p_e\sqrt{1 - \left(\frac{E'}{p_e c}\sin\theta\right)^2}$$

or

$$E - E'\cos\theta = \sqrt{p_e^2c^2 - (E'\sin\theta)^2}.$$

$$\Longrightarrow p_e^2c^2 = E^2 - 2EE'\cos\theta + E'^2\cos^2\theta + E'^2\sin^2\theta$$

$$\Longrightarrow p_e^2c^2 = E^2 - 2EE'\cos\theta + E'^2$$

Conservation of energy:

$$E + mc^2 = E' + \sqrt{m^2c^4 + p_e^2c^2} = E' + \sqrt{m^2c^4 + E^2 - 2EE'\cos\theta + E'^2}$$

$$\begin{aligned}(E - E' + mc^2)^2 &= \cancel{E^2} + \cancel{E'^2} + \cancel{m^2c^4} - 2EE' + 2Emc^2 - 2E'mc^2 \\ &= \cancel{m^2c^4} + E^2 - 2EE'\cos\theta + \cancel{E'^2}\end{aligned}$$

$$\cancel{2}mc^2(E - E') = \cancel{2}EE'(1-\cos\theta); \quad E' = \frac{hc}{\lambda'}, \quad E = \frac{hc}{\lambda}$$

$$\Longrightarrow m\cancel{c^2}hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h^2\cancel{c^2}}{\lambda\lambda'}(1-\cos\theta)$$

$$mc\left(\frac{\lambda' - \lambda}{\cancel{\lambda\lambda'}}\right) = \frac{h}{\cancel{\lambda\lambda'}}(1-\cos\theta) \Longrightarrow \boxed{\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta)}$$

4
Symmetries

Problem 4.1

movement of corners	symmetry operation
$A \to A, \quad B \to B, \quad C \to C$	I
$A \to A, \quad B \to C, \quad C \to B$	R_a
$A \to B, \quad B \to A, \quad C \to C$	R_c
$A \to B, \quad B \to C, \quad C \to A$	R_-
$A \to C, \quad B \to B, \quad C \to A$	R_b
$A \to C, \quad B \to A, \quad C \to B$	R_+

Problem 4.2

	I	R_+	R_-	R_a	R_b	R_c
I	I	R_+	R_-	R_a	R_b	R_c
R_+	R_+	R_-	I	R_b	R_c	R_a
R_-	R_-	I	R_+	R_c	R_a	R_b
R_a	R_a	R_c	R_b	I	R_-	R_+
R_b	R_b	R_a	R_c	R_+	I	R_-
R_c	R_c	R_b	R_a	R_-	R_+	I

The group is not Abelian; the multiplication table is not symmetrical across the main diagonal (for example, $R_+R_a = R_b$, but $R_aR_+ = R_c$).

Problem 4.3

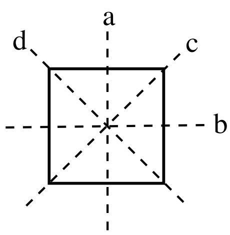
(a)

$$D(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D(R_-) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$D(R_a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad D(R_b) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad D(R_c) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) I, R_+ and R_- are represented by 1; R_a, R_b, and R_c are represented by -1. This representation is *not* faithful.

Problem 4.4



The square has $\boxed{8}$ symmetry operations: doing nothing (I), clockwise rotation through 90° (R_+), rotation through 180° (R_π), counterclockwise rotation through 90° (R_-), and flipping about the vertical axis a (R_a), the horizontal axis b (R_b), or the diagonal axes c (R_c) or d (R_d).

	I	R_+	R_π	R_-	R_a	R_b	R_c	R_d
I	I	R_+	R_π	R_-	R_a	R_b	R_c	R_d
R_+	R_+	R_π	R_-	I	R_c	R_d	R_b	R_a
R_π	R_π	R_-	I	R_+	R_b	R_a	R_d	R_c
R_-	R_-	I	R_+	R_π	R_d	R_c	R_a	R_b
R_a	R_a	R_d	R_b	R_c	I	R_π	R_-	R_+
R_b	R_b	R_c	R_a	R_d	R_π	I	R_+	R_-
R_c	R_c	R_a	R_d	R_b	R_+	R_-	I	R_π
R_d	R_d	R_b	R_c	R_a	R_-	R_+	R_π	I

The group is $\boxed{\text{not Abelian}}$ (for example, $R_+R_a = R_c$, but $R_aR_+ = R_d$).

Problem 4.5

(a) Let A and B be two $n \times n$ unitary matrices. Then

$$(\widetilde{AB})^*(AB) = \tilde{B}^*(\tilde{A}^*A)B = \tilde{B}^*B = 1,$$

so AB is also unitary (the set of $n \times n$ unitary matrices is closed under multiplication). The usual matrix identity is unitary. All unitary matrices have inverses ($A^{-1} = \tilde{A}^*$), which are themselves unitary. Finally, matrix multiplication is associative, so $U(n)$ a group.

(b) Suppose that the unitary matrices A and B both have determinant 1. Then $\det(AB) = \det(A)\det(B) = 1$. We already know that AB is unitary, so the set of $n \times n$ unitary matrices with determinant 1 is closed. The matrix identity has determinant 1, and $\det(A^{-1}) = 1/\det(A) = 1$, so the set contains the appropriate inverse and identity elements; $SU(n)$ is a group.

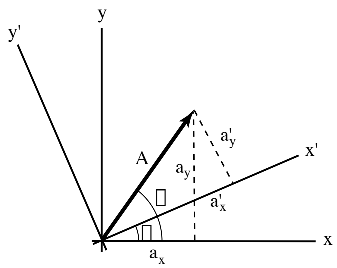
(c) $O(n)$, the real subset of $U(n)$ is closed by the same argument that $U(n)$ is closed:

$$(\widetilde{AB})(AB) = \tilde{B}(\tilde{A}A)B = \tilde{B}B = 1.$$

The identity, inverse, and associativity requirements are still met, so $O(n)$ is a group.

(d) The set of $n \times n$ orthogonal matrices is closed, as is the set of $n \times n$ matrices with determinant 1. Therefore, $SO(n)$ is closed. If $A \in SO(n)$, so is A^{-1}, and $I \in SO(n)$; $SO(n)$ is a group.

Problem 4.6



$$a_x = a\cos\phi; \quad a_y = a\sin\phi \quad (a = \text{length of } \mathbf{A})$$

$$a'_x = a\cos(\phi - \theta) = a(\cos\phi\cos\theta + \sin\phi\sin\theta) = \cos\theta a_x + \sin\theta a_y$$

$$a'_y = a\sin(\phi - \theta) = a(\sin\phi\cos\theta - \cos\phi\sin\theta) = \cos\theta a_y - \sin\theta a_x$$

$$\therefore \begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}. \qquad \boxed{R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.}$$

$$\tilde{R}R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} (\cos^2\theta + \sin^2\theta) & (\cos\theta\sin\theta - \sin\theta\cos\theta) \\ (\sin\theta\cos\theta - \cos\theta\sin\theta) & (\sin^2\theta + \cos^2\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so R is orthogonal. ✓

$$\boxed{\det(R) = \cos^2\theta + \sin^2\theta = 1} \quad \text{Group is } \boxed{SO(2).}$$

$$R(\theta_1)R(\theta_2)$$
$$= \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$$
$$= \begin{pmatrix} (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) & (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ (-\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) & (-\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} = R(\theta_1+\theta_2) = R(\theta_2+\theta_1)$$
$$= R(\theta_2)R(\theta_1).$$

Yes, it *is* Abelian.

Problem 4.7

$$\tilde{M}M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Yes, it \textit{is} in } O(2).$$

$\det(M) = -1.$ No, it is *not* in $SO(2)$.

$$M\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} a_x \\ -a_y \end{pmatrix}, \text{ so } \boxed{a'_x = a_x; \quad a'_y = -a_y.}$$

No, this is *not* a possible rotation of the xy plane. (It's a *reflection* in the x axis.)

Problem 4.8

For a spinning solid sphere, $I = \frac{2}{5}mr^2$ and $\omega = v/r$, so

$$\frac{1}{2}\hbar = L = I\omega = \left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right), \qquad v = \frac{5}{4}\frac{\hbar}{mr}.$$

An electron has mass $m = 9 \times 10^{-31}$ kg and radius $r < 10^{-18}$ m, giving $\boxed{v > 10^{14}\ \mathrm{m/s},}$ *much* faster than the speed of light. Evidently this classical model cannot be taken literally.

Problem 4.9

(a) *Before*: $2 \times 2 \times 2 = 8$ states. $(\uparrow\uparrow\uparrow, \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, \uparrow\downarrow\downarrow, \downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow, \downarrow\downarrow\uparrow, \downarrow\downarrow\downarrow)$
After: $s = \frac{3}{2}$ is 4 states; $s = \frac{1}{2}$ is 2 states, and occurs twice, so we get $4 + 2 + 2 = 8$. ✓

(b) 2 and 1 $\longrightarrow$ 3, 2 or 1.

$$\begin{aligned} 3 \text{ and } 1/2 &\longrightarrow 7/2 \quad (8 \text{ states}) \\ &\longrightarrow 5/2 \quad (6 \text{ states}) \\ 2 \text{ and } 1/2 &\longrightarrow 5/2 \quad (6 \text{ states}) \\ &\longrightarrow 3/2 \quad (4 \text{ states}) \\ 1 \text{ and } 1/2 &\longrightarrow 3/2 \quad (4 \text{ states}) \\ &\longrightarrow 1/2 \quad \underline{(2 \text{ states})} \\ & \qquad 8+6+6+4+4+2 = \boxed{30 \text{ states in all}} \end{aligned}$$

$\boxed{\text{We get total angular momenta } \tfrac{7}{2}, \tfrac{5}{2} \text{ (twice)}, \tfrac{3}{2} \text{ (twice), and } \tfrac{1}{2}.}$

We had $5 \times 3 \times 2 = 30$ states to begin with, and 30 at the end. ✓

Problem 4.10

Combining the spins of p and e, we can get $s = 1$ or $s = 0$, but not $s = \frac{1}{2}$. So this process would violate conservation of angular momentum. (If the p–e system carries *orbital* angular momentum we could achieve $s > 1$, but still only *integer* values: $\frac{1}{2}$ remains inaccessible.)

$\boxed{\tfrac{1}{2} \text{ would do}}$, for combining $s = 1$ with $s = \frac{1}{2}$ yields $\frac{3}{2}$ and $\frac{1}{2}$ (and combining $s = 0$ with $s = \frac{1}{2}$ yields $\frac{1}{2}$). But $\boxed{\tfrac{3}{2} \text{ would also be OK}}$, since combining $s = 1$ with $s = \frac{3}{2}$ yields $\frac{5}{2}, \frac{3}{2}$, and $\frac{1}{2}$. If we allow for orbital angular momentum in the final state, $\boxed{\text{any half-integer spin}}$ would work. But $\frac{1}{2}$ seems the most likely possibility.

Problem 4.11

The proton has spin $\frac{1}{2}$ and the pion has spin 0, so the total *spin* in the final state is $\frac{1}{2}$. This must combine with *orbital* angular momentum to give $s_\Delta = \frac{3}{2}$. Thus l could be $\boxed{1}$ $\left[(s=\frac{1}{2})+(l=1) \Rightarrow \frac{3}{2} \text{ or } \frac{1}{2}\right]$ or $\boxed{2}$ $\left[(s=\frac{1}{2})+(l=2) \Rightarrow \frac{5}{2} \text{ or } \frac{3}{2}\right]$.

Problem 4.12

From the $\frac{1}{2} \times 1$ Clebsch–Gordan table, I read

$$|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|1\ 0\rangle \underbrace{|\tfrac{1}{2}\ \tfrac{1}{2}\rangle}_{\text{spin up}}.$$

Probability of spin up is $\boxed{2/3.}$

Problem 4.13

From the 2×2 table, I read

$$|2\ 0\rangle|2\ 0\rangle = \sqrt{\tfrac{18}{35}}|4\ 0\rangle + 0|3\ 0\rangle + \left(-\sqrt{\tfrac{2}{7}}\right)|2\ 0\rangle + 0|1\ 0\rangle + \sqrt{\tfrac{1}{5}}|0\ 0\rangle.$$

So you could get

$$\boxed{j=4, \text{ probability } \tfrac{18}{35}; \quad j=2, \text{ probability } \tfrac{2}{7}; \quad \text{or } j=0, \text{ probability } \tfrac{1}{5}.}$$

(What this means, of course, is that the total angular momentum squared could come out $4(5)\hbar^2 = 20\hbar^2 (j=4)$ or $2(3)\hbar^2 = 6\hbar^2 (j=2)$ or $0 (j=0)$.) Do the probabilities add to 1?

$$\frac{18}{35} + \frac{2}{7} + \frac{1}{5} = \frac{18+10+7}{35} = \frac{35}{35} = 1. \checkmark$$

Problem 4.14

From the $2 \times \frac{3}{2}$ table, I read

$$|\tfrac{5}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{27}{70}}|2\ 1\rangle|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle + \sqrt{\tfrac{3}{35}}|2\ 0\rangle|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \left(-\sqrt{\tfrac{5}{14}}\right)|2\ -1\rangle|\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \left(-\sqrt{\tfrac{6}{35}}\right)|2\ -2\rangle|\tfrac{3}{2}\ \tfrac{3}{2}\rangle.$$

So you could get

$$\boxed{\begin{array}{l} m_1 = 1, \text{ probability } 27/70; \quad \text{or } 0, \text{ probability } 3/35; \\ \text{or } -1, \text{ probability } 5/14; \quad \text{or } -2, \text{ probability } 6/35. \end{array}}$$

(That is, S_z for the spin-2 particle could be $\hbar$, 0, $-\hbar$, or $-2\hbar$.) Do the probabilities add to 1?

$$\frac{27}{70} + \frac{3}{35} + \frac{5}{14} + \frac{6}{35} = \frac{27 + 6 + 25 + 12}{70} = \frac{70}{70} = 1. \checkmark$$

Problem 4.15

$$\hat{S}_x \chi_+ = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{\hbar}{2} \chi_+; \quad \text{eigenvalue } \boxed{\frac{\hbar}{2}.}$$

$$\hat{S}_x \chi_- = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = -\frac{\hbar}{2} \chi_-; \quad \text{eigenvalue } \boxed{-\frac{\hbar}{2}.}$$

Problem 4.16

$$\begin{aligned} |a|^2 + |b|^2 &= \left| \frac{1}{\sqrt{2}} (\alpha + \beta) \right|^2 + \left| \frac{1}{\sqrt{2}} (\alpha - \beta) \right|^2 \\ &= \frac{1}{2} \left[(\alpha^* + \beta^*) (\alpha + \beta) + (\alpha^* - \beta^*) (\alpha - \beta) \right] \\ &= \frac{1}{2} \left(|\alpha|^2 + \alpha^* \beta + \beta^* \alpha + |\beta|^2 + |\alpha|^2 - \alpha^* \beta - \beta^* \alpha + |\beta|^2 \right) \\ &= |\alpha|^2 + |\beta|^2 = 1. \quad \checkmark \end{aligned}$$

Problem 4.17

(a)

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Longrightarrow \begin{pmatrix} -i(\hbar/2)\beta \\ i(\hbar/2)\alpha \end{pmatrix} = \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix}$$

$$-i\frac{\hbar}{2}\beta = \lambda\alpha; \quad i\frac{\hbar}{2}\alpha = \lambda\beta \Rightarrow -i\frac{\hbar}{2}\beta = \lambda\left(\frac{2}{i\hbar}\lambda\beta\right) \Rightarrow \frac{\hbar^2}{4}\beta = \lambda^2\beta \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

Plus sign:

$$-i\frac{\hbar}{2}\beta = \frac{\hbar}{2}\alpha \Rightarrow \beta = i\alpha \Rightarrow \chi_+ = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ i\alpha\end{pmatrix}.$$

Minus sign:

$$-i\frac{\hbar}{2}\beta = -\frac{\hbar}{2}\alpha \Rightarrow \beta = -i\alpha \Rightarrow \chi_- = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix} = \begin{pmatrix}\alpha\\ -i\alpha\end{pmatrix}.$$

In both cases normalization ($|\alpha|^2 + |\beta|^2 = 1$) gives $\alpha = 1/\sqrt{2}$. *Conclusion:*

$\boxed{\text{Eigenvalues are } \pm\hbar/2.}$ $\boxed{\text{Eigenspinors are } \chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}.}$

(b)

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = c\chi_+ + d\chi_- = c\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + d\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}(c+d)\\ \frac{i}{\sqrt{2}}(c-d)\end{pmatrix}$$

$$\alpha = \frac{1}{\sqrt{2}}(c+d)\ \beta = \frac{i}{\sqrt{2}}(c-d) \Rightarrow \sqrt{2}(\alpha - i\beta) = 2c \Rightarrow c = \frac{1}{\sqrt{2}}(\alpha - i\beta)$$

$$\text{or}\quad \sqrt{2}(\alpha + i\beta) = 2d \Rightarrow d = \frac{1}{\sqrt{2}}(\alpha + i\beta)$$

You could get $\boxed{\pm\frac{1}{2}\hbar, \text{ probability } \frac{1}{2}|\alpha \mp i\beta|^2.}$

Problem 4.18

(a) From Eq. 4.24, probability of $\frac{1}{2}\hbar$ is

$$\left|\frac{1}{\sqrt{2}}(\alpha+\beta)\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right)\right|^2 = \boxed{\frac{9}{10}.}$$

Probability of $-\frac{1}{2}\hbar$ is

$$\left|\frac{1}{\sqrt{2}}(\alpha-\beta)\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right)\right|^2 = \boxed{\frac{1}{10}.}$$

(b) From Problem 4.17, probability of $\frac{1}{2}\hbar$ is

$$\left|\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}\right)\right|^2 = \frac{1}{10}(1-2i)(1+2i) = \frac{1}{10}(1+4) = \boxed{\frac{1}{2}.}$$

Probability of $-\frac{1}{2}\hbar$ is

$$\left|\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{5}} + \frac{2i}{\sqrt{5}}\right)\right|^2 = \boxed{\frac{1}{2}.}$$

(c) Probability of $\frac{1}{2}\hbar$ is $|\alpha|^2 = \boxed{\frac{1}{5}}$; probability of $-\frac{1}{2}\hbar$ is $|\beta|^2 = \boxed{\frac{4}{5}}$.

Problem 4.19

(a)

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

(b)

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z$$

$$\sigma_y\sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z$$

$$\sigma_y\sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x$$

$$\sigma_z\sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_x$$

$$\sigma_z\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_y$$

$$\sigma_x\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y$$

Problem 4.20

(a)

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = \delta_{ij} + i\epsilon_{ijk}\sigma_k - \delta_{ji} - i\epsilon_{jik}\sigma_k$$

But $\delta_{ij} = \delta_{ji}, \quad \epsilon_{jik} = -\epsilon_{ijk}$. So $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$.

(b)

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = \delta_{ij} + i\epsilon_{ijk}\sigma_k + \delta_{ji} + i\epsilon_{jik}\sigma_k = 2\delta_{ij}$$

(c)

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) &= \sum_{i,j}\sigma_i a_i \sigma_j b_j = \sum_{i,j} a_i b_j(\sigma_i\sigma_j) = \sum_{i,j} a_i b_j(\delta_{ij} + i\epsilon_{ijk}\sigma_k)\\ &= \sum_{i,j} a_i b_j \delta_{ij} + i\sum_{i,j}\epsilon_{ijk} a_i b_j \sigma_k = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})\end{aligned}$$

Problem 4.21

(a) Noting that $\sigma_z^2 = 1,\ \sigma_z^3 = \sigma_z$, etc.,

$$\begin{aligned}e^{i\theta\sigma_z} &= 1 + i\theta\sigma_z + \frac{(i\theta\sigma_z)^2}{2} + \frac{(i\theta\sigma_z)^3}{3!} + \cdots\\ &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots\right) + i\sigma_z\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\\ &= \cos\theta + i\sigma_z\sin\theta\end{aligned}$$

In particular $e^{i\pi\sigma_z/2} = \cos(\pi/2) + i\sigma_z\sin(\pi/2) = i\sigma_z$.

(b) Similarly (all we need is $\sigma_y^2 = 1$) $e^{i\theta\sigma_y} = \cos\theta + i\sigma_y\sin\theta$, so

$$U = \cos 90^\circ - i\sin 90^\circ\sigma_y = -i\sigma_y = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \boxed{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}$$

$$U\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix},\quad \text{which is spin down.}$$

(c)

$$U(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2} = 1 + \left(-i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}\right) + \frac{1}{2}\left(-i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}\right)^2 + \frac{1}{3!}\left(-i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}\right)^3 + \cdots$$

But $(\boldsymbol{\theta}\cdot\boldsymbol{\sigma})^2 = \boldsymbol{\theta}\cdot\boldsymbol{\theta} + i\boldsymbol{\sigma}\cdot(\boldsymbol{\theta}\times\boldsymbol{\sigma}) = \theta^2$. So

$$\begin{aligned}U(\boldsymbol{\theta}) &= 1 - i\frac{(\boldsymbol{\theta}\cdot\boldsymbol{\sigma})}{2} - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 + \frac{1}{3!}i(\boldsymbol{\theta}\cdot\boldsymbol{\sigma})\frac{\theta^2}{2^3} + \cdots\\ &= \left[1 - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - \cdots\right] - i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{\theta}\left[\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \cdots\right]\\ &= \cos\left(\frac{\theta}{2}\right) - i(\hat{\theta}\cdot\boldsymbol{\sigma})\sin\left(\frac{\theta}{2}\right).\end{aligned}$$

Problem 4.22

(a)

$$U = \cos\frac{\theta}{2} - i(\hat{\theta}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}; \quad U^\dagger = \cos\frac{\theta}{2} + i(\hat{\theta}\cdot\boldsymbol{\sigma}^\dagger)\sin\frac{\theta}{2}$$

But $\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}$ (the Pauli matrices are Hermitian), so

$$U^\dagger = \cos\frac{\theta}{2} + i(\hat{\theta}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}.$$

$$\begin{aligned}
UU^\dagger &= \left[\cos\frac{\theta}{2} - i(\hat{\theta}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}\right]\left[\cos\frac{\theta}{2} + i(\hat{\theta}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}\right] \\
&= \cos^2\frac{\theta}{2} - i(\hat{\theta}\cdot\boldsymbol{\sigma})\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\
&\quad + i(\hat{\theta}\cdot\boldsymbol{\sigma})\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right) + (\hat{\theta}\cdot\boldsymbol{\sigma})(\hat{\theta}\cdot\boldsymbol{\sigma})\sin^2\frac{\theta}{2} \\
&= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1. \quad \checkmark
\end{aligned}$$

[Note: I used Problem 4.20 **(c)** to show that

$$(\hat{\theta}\cdot\boldsymbol{\sigma})(\hat{\theta}\cdot\boldsymbol{\sigma}) = \hat{\theta}\cdot\hat{\theta} + i\boldsymbol{\sigma}\cdot(\hat{\theta}\times\hat{\theta}) = 1.]$$

(b)

$$\hat{\theta}\cdot\boldsymbol{\sigma} = \hat{\theta}_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \hat{\theta}_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \hat{\theta}_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\hat{\theta}_z & (\hat{\theta}_x - i\hat{\theta}_y)\\ (\hat{\theta}_x + i\hat{\theta}_y) & -\hat{\theta}_z\end{pmatrix}$$

$$\begin{aligned}
U(\boldsymbol{\theta}) &= \cos\frac{\theta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\sin\frac{\theta}{2}\begin{pmatrix}\hat{\theta}_z & (\hat{\theta}_x - i\hat{\theta}_y)\\ (\hat{\theta}_x + i\hat{\theta}_y) & -\hat{\theta}_z\end{pmatrix} \\
&= \begin{pmatrix}(\cos\frac{\theta}{2} - i\hat{\theta}_z\sin\frac{\theta}{2}) & -i\sin\frac{\theta}{2}(\hat{\theta}_x - i\hat{\theta}_y)\\ -i\sin\frac{\theta}{2}(\hat{\theta}_x + i\hat{\theta}_y) & (\cos\frac{\theta}{2} + i\hat{\theta}_z\sin\frac{\theta}{2})\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\det U &= \left(\cos\frac{\theta}{2} - i\hat{\theta}_z\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + i\hat{\theta}_z\sin\frac{\theta}{2}\right) + \sin^2\frac{\theta}{2}(\hat{\theta}_x + i\hat{\theta}_y)(\hat{\theta}_x - i\hat{\theta}_y) \\
&= \cos^2\frac{\theta}{2} + \hat{\theta}_z^2\sin^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}(\hat{\theta}_x^2 + \hat{\theta}_y^2) = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}(\hat{\theta}_x^2 + \hat{\theta}_y^2 + \hat{\theta}_z^2) \\
&= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1. \quad \checkmark
\end{aligned}$$

Problem 4.23

(a) Call the three column vectors χ_+, χ_0, χ_-. We have:

$$\hat{S}_z\chi_+ = \hbar\chi_+; \quad \hat{S}_z\chi_0 = 0; \quad \hat{S}_z\chi_- = -\hbar\chi_-; \quad \text{so} \quad \boxed{\hat{S}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}}.$$

(b)

$$\hat{S}_+|1\,1\rangle = 0; \quad \hat{S}_+|1\,0\rangle = \hbar\sqrt{2-0}|1\,1\rangle = \sqrt{2}\hbar|1\,1\rangle;$$

$$\hat{S}_+|1\,-1\rangle = \hbar\sqrt{2-0}|1\,0\rangle = \sqrt{2}\hbar|1\,0\rangle.$$

$$\therefore \hat{S}_+\begin{pmatrix}1\\0\\0\end{pmatrix} = 0; \quad \hat{S}_+\begin{pmatrix}0\\1\\0\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \hat{S}_+\begin{pmatrix}0\\0\\1\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

$$\therefore \boxed{\hat{S}_+ = \begin{pmatrix}0 & \sqrt{2}\hbar & 0\\ 0 & 0 & \sqrt{2}\hbar\\ 0 & 0 & 0\end{pmatrix}}.$$

$$\hat{S}_-|1\,1\rangle = \hbar\sqrt{2+0}|1\,0\rangle = \sqrt{2}\hbar|1\,0\rangle;$$

$$\hat{S}_-|1\,0\rangle = \hbar\sqrt{2+0}|1\,-1\rangle = \sqrt{2}\hbar|1\,-1\rangle; \quad \hat{S}_-|1\,-1\rangle = 0.$$

$$\therefore \boxed{\hat{S}_- = \begin{pmatrix}0 & 0 & 0\\ \sqrt{2}\hbar & 0 & 0\\ 0 & \sqrt{2}\hbar & 0\end{pmatrix}}.$$

(c)

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) = \boxed{\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}}; \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) = \boxed{\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}}$$

(d) Represent the four spin $\frac{3}{2}$ states ($m_s = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$) by the column vectors

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

$$\hat{S}_z|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = \frac{3}{2}\hbar|\tfrac{3}{2}\ \tfrac{3}{2}\rangle; \qquad \hat{S}_z|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \frac{1}{2}\hbar|\tfrac{3}{2}\ \tfrac{1}{2}\rangle;$$

$$\hat{S}_z|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = -\frac{1}{2}\hbar|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle; \qquad \hat{S}_z|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = -\frac{3}{2}\hbar|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle$$

$$\Longrightarrow \boxed{\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}}.$$

$$\hat{S}_+|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = 0; \qquad \hat{S}_+|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} - \tfrac{3}{4}}|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = \sqrt{3}\hbar|\tfrac{3}{2}\ \tfrac{3}{2}\rangle;$$

$$\hat{S}_+|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} + \tfrac{1}{4}}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = 2\hbar|\tfrac{3}{2}\ \tfrac{1}{2}\rangle;$$

$$\hat{S}_+|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} - \tfrac{3}{4}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \sqrt{3}\hbar|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle;$$

$$\Longrightarrow \boxed{\hat{S}_+ = \begin{pmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{pmatrix}}.$$

$$\hat{S}_-|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} - \tfrac{3}{4}}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{3}\hbar|\tfrac{3}{2}\ \tfrac{1}{2}\rangle;$$

$$\hat{S}_-|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} + \tfrac{1}{4}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = 2\hbar|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle;$$

$$\hat{S}_-|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} - \tfrac{3}{4}}|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = \sqrt{3}\hbar|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle; \qquad \hat{S}_-|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = 0$$

$$\Longrightarrow \boxed{\hat{S}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{pmatrix}}.$$

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) = \boxed{\frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}};$$

$$\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) = \boxed{\frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}}.$$

Problem 4.24

$\Omega^- = |0\,0\rangle; \quad \Sigma^+ = |1\,1\rangle; \quad \Xi^0 = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle; \quad \rho^+ = |1\,1\rangle; \quad \eta = |0\,0\rangle; \quad \bar{K}^0 = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle.$

Problem 4.25

(a)

$$u: \quad Q = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} + 0\right) = \frac{2}{3} \quad \checkmark$$

$$d: \quad Q = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} + 0\right) = -\frac{1}{3} \quad \checkmark$$

$$s: \quad Q = 0 + \frac{1}{2}\left(\frac{1}{3} - 1\right) = -\frac{1}{3} \quad \checkmark$$

(b)

$$\bar{u} = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle; \quad Q = -\frac{1}{2} + \frac{1}{2}\left(-\frac{1}{3} + 0\right) = -\frac{2}{3} \quad \checkmark$$

$$\bar{d} = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle; \quad Q = \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{3} + 0\right) = \frac{1}{3} \quad \checkmark$$

$$\bar{s} = |0\,0\rangle; \quad Q = 0 + \frac{1}{2}\left(-\frac{1}{3} + 1\right) = \frac{1}{3} \quad \checkmark$$

Problem 4.26

(a)

$$\boxed{Q = I_3 + \frac{1}{2}(A + S + C + B + T)}$$

(b)

$$\boxed{I_3 = \frac{1}{2}(U + D)}; \quad \boxed{A = \frac{1}{3}(U + C + T - D - S - B)}$$

(c)

$$Q = \frac{1}{2}(U+D) + \frac{1}{2}\left[\frac{1}{3}(U+C+T-D-S-B)+S+C+B+T\right]$$
$$= \boxed{\frac{1}{3}\left[2(U+B+T)+(D+S+B)\right]}$$

Problem 4.27

$I_{\text{tot}}^2 = (\mathbf{I}_1+\mathbf{I}_2)^2 = I_1^2 + I_2^2 + 2\mathbf{I}_1\cdot\mathbf{I}_2$. Here $I_1^2 = I_2^2 = \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}$.

$I_{\text{tot}}^2 = 0$ in singlet state, so $0 = \frac{3}{4}+\frac{3}{4}+2\mathbf{I}_1\cdot\mathbf{I}_2$, or $\boxed{\mathbf{I}_1\cdot\mathbf{I}_2 = -\frac{3}{4}.}$

$I_{\text{tot}}^2 = 1(1+1) = 2$ in triplet, so $2 = \frac{3}{4}+\frac{3}{4}+2\mathbf{I}_1\cdot\mathbf{I}_2$, or $\boxed{\mathbf{I}_1\cdot\mathbf{I}_2 = \frac{1}{4}.}$

Problem 4.28

(a)

$$\mathcal{M}_a = \mathcal{M}_f = \mathcal{M}_3\,;$$
$$\mathcal{M}_b = \mathcal{M}e = \frac{2}{3}\mathcal{M}_3 + \frac{1}{3}\mathcal{M}_1\,;$$
$$\mathcal{M}_c = \mathcal{M}_d = \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1\,;$$
$$\mathcal{M}_g = \mathcal{M}_h = \mathcal{M}_i = \mathcal{M}_j = (\sqrt{2}/3)\mathcal{M}_3 - (\sqrt{2}/3)\mathcal{M}_1.$$

(b)

$$\sigma_a : \sigma_b : \sigma_c : \sigma_d : \sigma_e : \sigma_f : \sigma_g : \sigma_h : \sigma_i : \sigma_j =$$

$$\boxed{\begin{aligned}&9|\mathcal{M}_3|^2 : |2\mathcal{M}_3+\mathcal{M}_1|^2 : |\mathcal{M}_3+2\mathcal{M}_1|^2 : |\mathcal{M}_3+2\mathcal{M}_1|^2\\ &: |2\mathcal{M}_3+\mathcal{M}_1|^2 : 9|\mathcal{M}_3|^2 : 2|\mathcal{M}_3-\mathcal{M}_1|^2 : 2|\mathcal{M}_3-\mathcal{M}_1|^2\\ &: 2|\mathcal{M}_3-\mathcal{M}_1|^2 : 2|\mathcal{M}_3-\mathcal{M}_1|^2\end{aligned}}$$

(c)

$$\sigma_a : \sigma_b : \sigma_c : \sigma_d : \sigma_e : \sigma_f : \sigma_g : \sigma_h : \sigma_i : \sigma_j = \boxed{9:4:1:1:4:9:2:2:2:2}$$

Problem 4.29

Using the method in Section 4.3:

$$\begin{cases} \pi^- + p: & |1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \\ \pi^+ + p: & |1\ 1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle \end{cases}$$

$$\begin{cases} K^0 + \Sigma^0 & : & |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle|1\ 0\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \\ K^+ + \Sigma^- & : & |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|1\ -1\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \\ K^+ + \Sigma^+ & : & |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|1\ 1\rangle = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle \end{cases}$$

$$\mathcal{M}_c = \mathcal{M}_3; \quad \mathcal{M}_a = \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1; \quad \mathcal{M}_b = \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1$$

$$\therefore \sigma_a : \sigma_b : \sigma_c = \frac{2}{9}|\mathcal{M}_3 - \mathcal{M}_1|^2 : \frac{1}{9}|\mathcal{M}_3 + 2\mathcal{M}_1|^2 : |\mathcal{M}_3|^2.$$

If the $I = 3/2$ channel dominates, $\mathcal{M}_3 \gg \mathcal{M}_1$, so $\boxed{\sigma_a : \sigma_b : \sigma_c = 2 : 1 : 9.}$
If the $I = 1/2$ channel dominates, $\mathcal{M}_1 \gg \mathcal{M}_3$, so $\boxed{\sigma_a : \sigma_b : \sigma_c = 2 : 4 : 9.}$

Problem 4.30

$$\begin{cases} K^- + p: & |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}}|1\ 0\rangle - \sqrt{\tfrac{1}{2}}|0\ 0\rangle \\ \bar{K}^0 + p\ : & |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |1\ 1\rangle \end{cases}$$

$$\begin{cases} \Sigma^0 + \pi^0 & : & |1\ 0\rangle|1\ 0\rangle = \sqrt{\tfrac{2}{3}}|2\ 0\rangle - \sqrt{\tfrac{1}{3}}|0\ 0\rangle \\ \Sigma^+ + \pi^- & : & |1\ 1\rangle|1\ -1\rangle = \sqrt{\tfrac{1}{6}}|2\ 0\rangle + \sqrt{\tfrac{1}{2}}|1\ 0\rangle + \sqrt{\tfrac{1}{3}}|0\ 0\rangle \\ \Sigma^+ + \pi^0 & : & |1\ 1\rangle|1\ 0\rangle = \sqrt{\tfrac{1}{2}}|2\ 1\rangle + \sqrt{\tfrac{1}{2}}|1\ 1\rangle \\ \Sigma^0 + \pi^+ & : & |1\ 0\rangle|1\ 1\rangle = \sqrt{\tfrac{1}{2}}|2\ 1\rangle - \sqrt{\tfrac{1}{2}}|1\ 1\rangle \end{cases}$$

$$\begin{aligned} (\mathbf{a})\ \mathcal{M}_a &= -\sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{3}}\mathcal{M}_0; \quad \boxed{I_{\text{tot}} = 0} \\ (\mathbf{b})\ \mathcal{M}_b &= \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}}\mathcal{M}_1 + \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{3}}\mathcal{M}_0; \quad \boxed{I_{\text{tot}} = 0 \text{ or } 1} \\ (\mathbf{c})\ \mathcal{M}_c &= \sqrt{\tfrac{1}{2}}\mathcal{M}_1; \quad \boxed{I_{\text{tot}} = 1} \\ (\mathbf{d})\ \mathcal{M}_d &= -\sqrt{\tfrac{1}{2}}\mathcal{M}_1; \quad \boxed{I_{\text{tot}} = 1} \end{aligned}$$

$$\sigma_a : \sigma_b : \sigma_c : \sigma_d = \frac{1}{6}|\mathcal{M}_0|^2 \; : \; \frac{1}{2}|\sqrt{\tfrac{1}{3}}\mathcal{M}_0 + \sqrt{\tfrac{1}{2}}\mathcal{M}_1|^2 \; : \; \frac{1}{2}|\mathcal{M}_1|^2 \; : \; \frac{1}{2}|\mathcal{M}_1|^2.$$

If $I = 0$ dominates, $\boxed{\sigma_a : \sigma_b : \sigma_c : \sigma_d = 1 : 1 : 0 : 0}$
If $I = 1$ dominates, $\boxed{\sigma_a : \sigma_b : \sigma_c : \sigma_d = 0 : 1 : 2 : 2}$

Problem 4.31

If $\mathcal{M}_1 \gg \mathcal{M}_3$, Eq. 4.50 becomes $\sigma_a : \sigma_c : \sigma_j = 0 : 2 : 1$, so $\sigma_{\text{tot}}(\pi^+ + p) = 0$, $\sigma_{\text{tot}}(\pi^- + p) = 3$, and $\sigma_{\text{tot}}(\pi^+ + p)/\sigma_{\text{tot}}(\pi^- + p) = 0$. There is no sign of a resonance in $\sigma_{\text{tot}}(\pi^+ + p)$ at 1525, 1688, or 2190, so these must be $I = 1/2$. (The ratios are not zero, of course—evidently there is a lot of resonant background.) There *is* a clear resonance in $\sigma_{\text{tot}}(\pi^+ + p)$ at 1920; this must be $I = 3/2$. (The ratio is $\sigma_{\text{tot}}(\pi^+ + p)/\sigma_{\text{tot}}(\pi^- + p) = 43/36 = 1.2$, which is not very close to 3—presumably this is again due to nonresonant background.)

The nomenclature should be $\boxed{N(1525), N(1688), \Delta(1920), N(2190).}$ The Particle Physics Booklet lists $N(1520, N(1680), \Delta(1920), N(2190)$ (and there are others with roughly the same mass).

Problem 4.32

$$\left\{ \Sigma^{*0} : \quad |1\,0\rangle \right.$$

$$\left\{ \begin{array}{lll} (a)\ \Sigma^+ + \pi^- : & |1\,1\rangle|1\ -1\rangle = \sqrt{\tfrac{1}{6}}|2\,0\rangle + \sqrt{\tfrac{1}{2}}|1\,0\rangle + \sqrt{\tfrac{1}{3}}|0\,0\rangle \\ (b)\ \Sigma^0 + \pi^0 \ : & |1\,0\rangle|1\,0\rangle = \sqrt{\tfrac{2}{3}}|2\,0\rangle - \sqrt{\tfrac{1}{3}}|0\,0\rangle \\ (c)\ \Sigma^- + \pi^+ : & |1\ -1\rangle|1\,1\rangle = \sqrt{\tfrac{1}{6}}|2\,0\rangle - \sqrt{\tfrac{1}{2}}|1\,0\rangle + \sqrt{\tfrac{1}{3}}|0\,0\rangle \end{array} \right.$$

$$\mathcal{M}_a = \sqrt{\tfrac{1}{2}}\mathcal{M}_1, \quad \mathcal{M}_b = 0, \quad \mathcal{M}_c = -\sqrt{\tfrac{1}{2}}\mathcal{M}_1. \qquad \sigma_a : \sigma_b : \sigma_c = 1 : 0 : 1$$

I'd expect to see about 50 decays each to $\Sigma^+ + \pi^-$ and $\Sigma^- + \pi^+$, but none to $\Sigma^0 + \pi^0$.

Problem 4.33

(a) Isospin must be $\boxed{\text{zero.}}$

(b) The deuterons carry $I = 0$, so the isospin on the left is *zero*. The α has $I = 0$, and π has $I = 1$, so the isospin on the right is *one*. This process does not conserve isospin, and hence is not a possible strong interaction.

(c) There are five possible 4-nucleon states:

$$(nnnn),\ (nnnp) = {}^4\text{H},\ (nnpp) = {}^4\text{He},\ (nppp) = {}^4\text{Li},\ (pppp) = {}^4\text{Be}$$

In principle they could form an $I = 2$ multiplet, but since ^{4}H and ^{4}Li do not exist, this is out. $\boxed{\text{No: } {}^4\text{Be and } (nnnn) \text{ should } \textit{not} \text{ exist.}}$ (^{4}H, ^{4}He, and ^{4}Li could make an $I = 1$ multiplet, but, again, ^{4}H and ^{4}Li do not exist, so this too is out. Evidently four nucleons bind *only* in the $I = 0$ combination, making ^{4}He.)

Problem 4.34

(a) Suppose f is an eigenfunction of P: $Pf = \lambda f$. Then

$$P^2 f = P(Pf) = P(\lambda f) = \lambda(Pf) = \lambda(\lambda f) = \lambda^2 f.$$

But $P^2 = 1$, so $\lambda^2 = 1$, and hence $\lambda = \pm 1$. ✓
(b) Let $\boxed{f_\pm(x,y,z) \equiv \frac{1}{2}\left[f(x,y,z) \pm f(-x,-y,-z)\right]}$. Then $f = f_+ + f_-$. Note that

$$Pf_\pm = \tfrac{1}{2}\left[f(-x,-y,-z) \pm f(x,y,z)\right] = \pm f_\pm,$$

so $f_\pm$ is an eigenfunction of P, with eigenvalue ± 1.

Problem 4.35

(a) $\boxed{\text{No.}}$ P would transform a (left-handed) neutrino into a right-handed neutrino, which doesn't even exist (in the massless limit).
(b) The K has intrinsic parity -1, so the $\boxed{\textit{two}\text{-pion decay}}$ violates conservation of parity.

Problem 4.36

(a) Equation 4.60 says $G = (-1)^I C$, so

$$\begin{aligned}
\pi &: \; I = 1, C = 1 \;\; \Rightarrow \boxed{G = -1} \\
\rho &: \; I = 1, C = -1 \Rightarrow \boxed{G = 1} \\
\omega &: \; I = 0, C = -1 \Rightarrow \boxed{G = -1} \\
\eta &: \; I = 0, C = 1 \;\; \Rightarrow \boxed{G = 1} \\
\eta' &: \; I = 0, C = 1 \;\; \Rightarrow \boxed{G = 1} \\
\phi &: \; I = 0, C = -1 \Rightarrow \boxed{G = -1} \\
f_2 &: \; I = 0, C = 1 \;\; \Rightarrow \boxed{G = 1}
\end{aligned}$$

(b) [[?????]]

Problem 4.37

(a) Since the η and the π's have spin zero, the final state would need to have orbital angular momentum zero: $l = 0$ (by conservation of angular momentum). But this means that the parity of the final state is $(P_\pi)(P_\pi)(-1)^l = (-1)(-1)(-1)^0 = 1$. However, the parity of the η is -1, so this decay is *forbidden by conservation of parity*.

(b) The G-parity of the final state is $(-1)^3 = -1$ (Eq. 4.61). But the G-parity of the η is $(-1)^0(1) = +1$ (Eq. 4.60 and Table 4.6). So $\eta \longrightarrow 3\pi$ violates conservation of G-parity, and hence is a forbidden strong interaction.

Problem 4.38

Same baryon number as antiparticle $\Rightarrow$ meson; same charge as antiparticle $\Rightarrow$ neutral. We're not interested in $q\bar{q}$, since these are already their own antiparitlces; we need $q_1\bar{q}_2$, with $Q_1 = Q_2 = Q$:

$$Q = -1/3 : \boxed{d\bar{s} \leftrightarrow s\bar{d},\; d\bar{b} \leftrightarrow b\bar{d},\; s\bar{b} \leftrightarrow b\bar{s}.}$$

$$Q = 2/3 : \boxed{u\bar{c} \leftrightarrow c\bar{u}} \quad \text{(only, since } t \text{ forms no bound states)}.$$

So the candidate mesons are $\boxed{K^0 \leftrightarrow \bar{K}^0,\; B^0 \leftrightarrow \bar{B}^0,\; B_s^0 \leftrightarrow \bar{B}_s^0,\; D^0 \leftrightarrow \bar{D}^0.}$

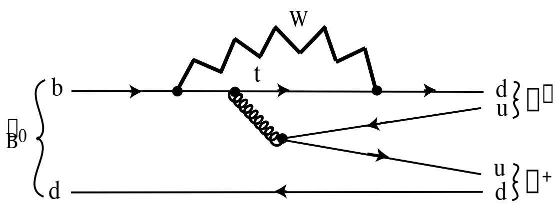
Neutron/antineutron oscillation would violate conservation of baryon number. The vector mesons decay by the strong interaction long before they would have a chance to interconvert.

Problem 4.39

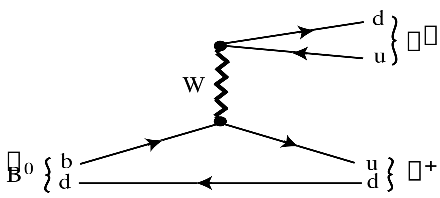
First establish the convention for positive charge, as in Section 4.4.3: it is the charge carried by the lepton preferentially produced in the decay of the long-lived neutral K meson. Then define right-handedness: it is the helicity of the charged lepton produced in the decay of a positively charged pion. Finally, define "up" as the direction away from the earth, and "front" as the side our eyes are on; cross "up" with "front", using the right-hand rule, and the result is the side our hearts are on. (Identifying the particle types is easy—just make a list in order of increasing mass.)

Problem 4.40

Penguin:



Tree:



5
Bound States

Problem 5.1

(a) $m_d - m_p - m_n = 1875.6 - 938.27 - 939.57 = -2.2$, so the binding energy is $\boxed{2.2\text{ MeV}}$, which is only 0.12% of the total. $\boxed{\text{No}}$, it's *not* relativistic.
(b) $m_\pi - m_d - m_u = 140 - 340 - 336 = -536$, so the binding energy is $\boxed{536\text{ MeV}}$, which is 3.8 *times* the total. $\boxed{\text{Yes}}$, it *is* relativistic.

Problem 5.2

Here $n = 1, l = 0, m_l = 0$, so

$$\Psi_{100} = \left[\left(\frac{2}{a}\right)^3 \frac{1}{2}\right]^{1/2} e^{-r/a}\, L_0^1(2r/a) Y_0^0(\theta, \phi)\, e^{-iE_1 t/\hbar}.$$

But $\quad L_0^1 = 1, \quad$ and $\quad Y_0^0 = 1/\sqrt{4\pi} \quad$ so

$$\Psi_{100} = \frac{2}{a^{3/2}} e^{-r/a} \frac{1}{2\sqrt{\pi}} e^{-iE_1 t/\hbar} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} e^{-iE_1 t/\hbar},$$

where $\quad E_1 = -me^4/2\hbar^2$.
Now $\quad \nabla^2 f(r) = (1/r^2)(d/dr)[r^2\, df/dr], \quad$ so

$$\begin{aligned}\nabla^2 \Psi_{100} &= \frac{1}{\sqrt{\pi a^3}} e^{-iE_1 t/\hbar} \frac{1}{r^2} \frac{d}{dr}\left[r^2\left(-\frac{1}{a}\right) e^{-r/a}\right] \\ &= \frac{1}{\sqrt{\pi a^3}} e^{-iE_1 t/\hbar} \left(-\frac{1}{a}\right) \frac{1}{r^2}\left[2re^{-r/a} - \frac{1}{a} r^2 e^{-r/a}\right] = \left(\frac{1}{a^2} - \frac{2}{ar}\right)\Psi_{100}.\end{aligned}$$

The left side of the Schrödinger equation is

$$lhs = -\frac{\hbar^2}{2m}\left(\frac{1}{a^2} - \frac{2}{ar}\right)\Psi_{100} - \frac{e^2}{r}\Psi_{100}.$$

But $a = \hbar^2/me^2$, so the second term cancels the third, leaving

$$lhs = -\frac{\hbar^2}{2ma^2}\Psi_{100} = -\frac{me^4}{2\hbar^2}\Psi_{100}.$$

Meanwhile, the right side is

$$i\hbar\frac{\partial\Psi_{100}}{\partial t} = i\hbar\left(-\frac{iE_1}{\hbar}\right)\Psi_{100} = E_1\Psi_{100} = -\frac{me^4}{2\hbar^2}\Psi_{100}. \quad \checkmark$$

Normalization requires

$$\int |\Psi|^2\, r^2\, \sin\theta\, dr\, d\theta\, d\phi = 1.$$

Here

$$|\Psi_{100}|^2 = \frac{1}{\pi a^3}e^{-2r/a}, \quad \text{and} \quad \int \sin\theta\, d\theta\, d\phi = 4\pi, \quad \text{so}$$

$$\int |\Psi_{100}|^2\, r^2\, \sin\theta\, dr\, d\theta\, d\phi = \frac{1}{\pi a^3}4\pi\int_0^\infty e^{-2r/a}r^2\, dr = \frac{4}{a^3}\left[2\left(\frac{a}{2}\right)^3\right] = 1. \quad \checkmark$$

Problem 5.3

There are four states:

$$\Psi_{200} = \left[\left(\frac{2}{2a}\right)^3\frac{1}{4(2)^3}\right]^{1/2} e^{-r/2a}\, L_1^1(2r/2a)\, Y_0^0\, e^{-iE_2t/\hbar}$$

$$= \boxed{\frac{1}{4a\sqrt{2\pi a}}\left(2-\frac{r}{a}\right)e^{-r/2a}e^{-iE_2t/\hbar}}$$

$$\Psi_{210} = \left[\left(\frac{2}{2a}\right)^3\frac{1}{4(6)^3}\right]^{1/2} e^{-r/2a}\left(\frac{2r}{2a}\right) L_0^3(2r/2a)\, Y_1^0\, e^{-iE_2t/\hbar}$$

$$= \boxed{\frac{1}{4a^2\sqrt{2\pi a}}re^{-r/2a}\cos\theta e^{-iE_2t/\hbar}}$$

$$\Psi_{21\pm1} = \left[\left(\frac{2}{2a}\right)^3\frac{1}{4(6)^3}\right]^{1/2} e^{-r/2a}\left(\frac{2r}{2a}\right) L_0^3(2r/2a)\, Y_1^{\pm1}\, e^{-iE_2t/\hbar}$$

$$= \boxed{\mp\frac{1}{8a^2\sqrt{\pi a}}re^{-r/2a}\sin\theta e^{\pm i\phi}e^{-iE_2t/\hbar}}$$

where $E_2 = -me^4/8\hbar^2$, and I used

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{\pm i\phi},$$

and $L_1^1(x) = -2x + 4,\ L_0^3 = 6.$

Problem 5.4

$$E = c\sqrt{\mathbf{p}^2 + m^2c^2} = mc^2\left(1 + \frac{\mathbf{p}^2}{m^2c^2}\right)^{1/2}.$$

Binomial expansion:

$$(1+\epsilon)^{1/2} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \cdots \Rightarrow$$

$$E \approx mc^2\left[1 + \frac{\mathbf{p}^2}{2m^2c^2} - \frac{(\mathbf{p}^2)^2}{8m^4c^4}\right] = mc^2 + \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}^4}{8m^3c^2}.$$

So

$$T = E - mc^2 \approx \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}^4}{8m^3c^2}. \quad \checkmark$$

Problem 5.5

From Eq. 5.19,

$$\begin{aligned} E_{3/2} - E_{1/2} &= \left[E_2 - \alpha^4 mc^2 \frac{1}{4(2)^4}\left(\frac{4}{2} - \frac{3}{2}\right)\right] - \left[E_2 - \alpha^4 mc^2 \frac{1}{4(2)^4}\left(\frac{4}{1} - \frac{3}{2}\right)\right] \\ &= \frac{1}{32}\alpha^4 mc^2 = \frac{1}{32}\left(\frac{1}{137}\right)^4 (0.511 \times 10^6) = \boxed{4.53 \times 10^{-5}\ \text{eV}}. \end{aligned}$$

By comparison,

$$E_2 - E_1 = E_1\left(\frac{1}{4} - 1\right) = -\frac{3}{4}E_1 = -\frac{3}{4}(-13.6) = \boxed{10.2\ \text{eV}}$$

so the fine structure is smaller by a factor of nearly a million.

Problem 5.6

From Eq. 5.20,

$$\Delta E_S = \alpha^5 mc^2 \frac{1}{4(2)^3} k(2,0) \approx \frac{13}{32}\alpha^5 mc^2$$

(using $k(2,0) \approx 13$). From Eq. 5.21,

$$\Delta E_P = \alpha^5 mc^2 \frac{1}{4(2)^3}\left[k(2,1) - \frac{1}{\pi(3/2)}\right] \approx -\frac{1}{32}\alpha^5 mc^2\left(\frac{2}{3\pi}\right)$$

(I dropped $k(2,1) < 0.05$ in comparison with $2/3\pi = 0.21$). So the Lamb shift is

$$\Delta E = \Delta E_S - \Delta E_P \approx \left(13 + \frac{2}{3\pi}\right)\frac{1}{32}\alpha^5 mc^2 = \frac{13.2}{32}\left(\frac{1}{137}\right)^5 (0.511\times 10^6)$$
$$= \boxed{4.37\times 10^{-6}\text{ eV}}.$$

$$E = h\nu \Rightarrow \nu = \frac{E}{h} = \frac{4.37\times 10^{-6}}{2\pi(6.58\times 10^{-16})} = \boxed{1.057\times 10^9\text{ Hz}}.$$

(Not bad!)

Problem 5.7

In the absence of *any* splitting there are 16 completely degenerate states (four orbital states, and for each of these two electron spin states and two proton spin states). Fine structure splits these into two distinct energy levels, one for $j = 1/2$ and one for $j = 3/2$ (the $2S_{1/2}$ and $2P_{1/2}$ states remain degenerate, $2P_{3/2}$ is slightly higher). The Lamb shift lifts the $2S_{1/2}/2P_{1/2}$ degeneracy, so there are now *three* distinct energy levels. Hyperfine splitting further separates each of these into two, according to the value of f (0 or 1, if $j = 1/2$; 1 or 2, if $j = 3/2$). So there are $\boxed{\text{6 energy levels}}$ in all: $n = 2, l = 0, j = 1/2, f = 0$ (one state); $n = 2, l = 0, j = 1/2, f = 1$ (three states); $n = 2, l = 1, j = 1/2, f = 0$ (one state); $n = 2, l = 1, j = 1/2, f = 1$ (three states); $n = 2, l = 1, j = 3/2, f = 1$ (three states); $n = 2, l = 1, j = 3/2, f = 2$ (five states)—still 16 states in all.

Using Eq. 5.23 with $n = 2, j = 1/2, f = 0$ (lower sign), and $l = 0, 1$:

$$\Delta E_S = \left(\frac{m}{m_p}\right)\alpha^4 mc^2 \frac{\gamma_p}{16}\frac{-1}{(1/2)(1/2)} = -\frac{1}{4}\left(\frac{m}{m_p}\right)\alpha^4 mc^2\gamma_p,$$

$$\Delta E_P = \left(\frac{m}{m_p}\right)\alpha^4 mc^2 \frac{\gamma_p}{16}\frac{-1}{(1/2)(3/2)} = -\frac{1}{12}\left(\frac{m}{m_p}\right)\alpha^4 mc^2\gamma_p.$$

$$\Delta E = \Delta E_S - \Delta E_P = -\frac{1}{6}\left(\frac{m}{m_p}\right)\alpha^4 mc^2\gamma_p$$
$$= -\frac{1}{6}\left(\frac{.511}{938}\right)\left(\frac{1}{137}\right)^4 (0.511\times 10^6)(2.79) = \boxed{3.67\times 10^{-7}\text{ eV}}.$$

This is about a *tenth* the Lamb shift.

Problem 5.8

For $n = 3$ we have $l = 0$ (S), $l = 1$ (P), and $l = 2$ (D). For $l = 0$ we can have either the singlet or the triplet spin configuration (since $l = 0$, $j = s$ automatically). For $l = 1$ we can have the singlet (with $j = 1$ automatically) or the tripet (for which j can be 0, 1, or 2). For $l = 2$ we have the singlet (with $j = 2$ automatic) or the triplet (for which j can be 1, 2, or 3). So there are 10 levels in all. But it turns out that $^3P_2$ and $^3D_2$ are degenerate, so in fact there are just $\boxed{9}$ distinct energy levels.

The *unperturbed* energy of all these states is (Eq. 5.27)

$$E_3 = -\alpha^2 mc^2 \frac{1}{36} = \frac{-13.6}{18} = -0.756 \text{ eV}.$$

The fine/hyperfine correction is (Eq. 5.29)

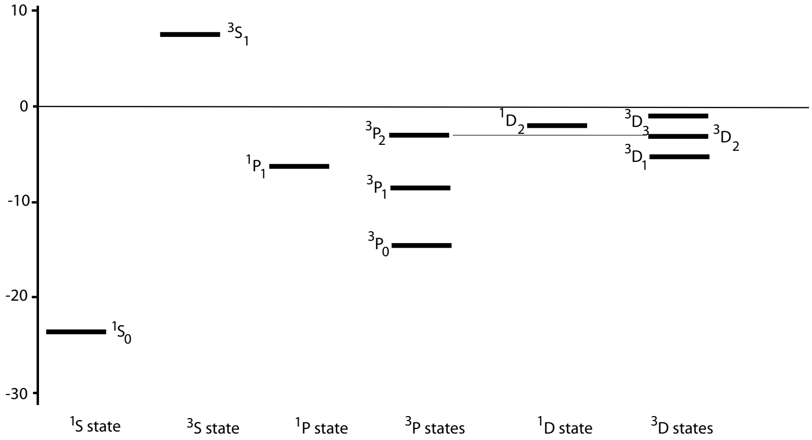
$$E_{\text{fs}} = \alpha^4 mc^2 \frac{1}{54}\left[\frac{11}{96} - \frac{(1+\epsilon/2)}{(2l+2)}\right] = (2.68 \times 10^{-5})\left[0.1146 - \frac{(1+\epsilon/2)}{(2l+1)}\right]$$

(with ϵ given by Eq. 5.30). In addition there is an annihilation correction, which applies only to the triplet S state (Eq. 5.31)

$$E_{\text{ann}} = \alpha^4 mc^2 \frac{1}{108} = 1.342 \times 10^{-5} \text{ eV}.$$

In units of μeV (10^{-6} eV), the corrections are:

$$\begin{array}{lll}
{}^1S_0: & l=0, s=0, j=0, \epsilon=0; & \Delta E = 26.8(0.1146-1) = \boxed{-23.7} \\
{}^3S_1: & l=0, s=1, j=1, \epsilon=-\frac{4}{3}; & \Delta E = 26.8(0.1146-\frac{1}{3}) + 13.42 = \boxed{7.56} \\
{}^1P_1: & l=1, s=0, j=1, \epsilon=0; & \Delta E = 26.8(0.1146-\frac{1}{3}) = \boxed{-5.86} \\
{}^3P_2: & l=1, s=1, j=2, \epsilon=-\frac{7}{10}; & \Delta E = 26.8(0.1146-\frac{13}{60}) = \boxed{-2.74} \\
{}^3P_1: & l=1, s=1, j=1, \epsilon=\frac{1}{2}; & \Delta E = 26.8(0.1146-\frac{5}{12}) = \boxed{-8.10} \\
{}^3P_0: & l=1, s=1, j=0, \epsilon=2; & \Delta E = 26.8(0.1146-\frac{2}{3}) = \boxed{-14.8} \\
{}^1D_2: & l=2, s=0, j=2, \epsilon=0; & \Delta E = 26.8(0.1146-\frac{1}{5}) = \boxed{-2.29} \\
{}^3D_3: & l=2, s=1, j=3, \epsilon=-\frac{10}{21}; & \Delta E = 26.8(0.1146-\frac{16}{105}) = \boxed{-1.01} \\
{}^3D_2: & l=2, s=1, j=2, \epsilon=\frac{1}{6}; & \Delta E = 26.8(0.1146-\frac{13}{60}) = \boxed{-2.74} \\
{}^3D_1: & l=2, s=1, j=1, \epsilon=\frac{5}{6}; & \Delta E = 26.8(0.1146-\frac{17}{60}) = \boxed{-4.52}
\end{array}$$



Problem 5.9

The ϕ is $\boxed{\text{quasi-bound}}$, because the OZI-allowed decay into a pair of K's is kinematically permissible ($m_\phi = 1019 > 2m_K = 990$).

Problem 5.10

The parameters we have to work with (in solving the Schrödinger equation) are

$$m \text{ (units of kg} = \text{J}\cdot\text{s}^2/\text{m}^2), \quad \hbar \text{ (units of J}\cdot\text{s)}, \quad F_0 \text{ (units of N} = \text{J/m)}.$$

From these we must construct an *energy*:

$$\text{J} = m^\alpha \hbar^\beta F_0^\gamma = \left(\frac{\text{J}\cdot\text{s}^2}{\text{m}^2}\right)^\alpha (\text{J}\cdot\text{s})^\beta \left(\frac{\text{J}}{\text{m}}\right)^\gamma = \text{J}^{\alpha+\beta+\gamma}\text{s}^{2\alpha+\beta}\text{m}^{-2\alpha-\gamma}.$$

Evidently

$$2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha;\ 2\alpha + \gamma = 0 \Rightarrow \gamma = -2\alpha;\ \alpha + \beta + \gamma = -3\alpha = 1,$$

so $\alpha = -1/3, \beta = \gamma = 2/3$, and hence

$$E = m^{-1/3}\hbar^{2/3}F_0^{2/3}a = \left[\frac{(\hbar F_0)^2}{m}\right]^{1/3} a,$$

where a is some numerical factor.

Problem 5.11

For the ψ's, $M = 2m_c + E_n = 2500 + E_n$:

F_0	M_1	M_2	M_3	M_4
500	2800	3200	3500	3700
1000	3000	3600	4100	4400
1500	3200	4000	4500	5100
Expt	3097	3686	4039	4159

For the Υ's, $M = 2m_b + E_n = 9000 + E_n$:

F_0	M_1	M_2	M_3	M_4
500	9300	9700	10000	10200
1000	9500	10100	10600	10900
1500	9700	10500	11000	11600
Expt	9460	10023	10355	10579

Level spacings:

F_0	$M_2 - M_1$	$M_3 - M_2$	$M_4 - M_3$
500	400	300	200
1000	600	500	400
1500	800	600	500
Expt (ψ)	589	353	120
Expt (Υ)	563	332	224

Evidently F_0 is between 500 and 1000; of the three choices, $\boxed{500\text{ MeV/fm}}$ is best (though 1000 looks better for the masses themselves). The results are not exact, of course, because (a) the potential (Eq. 5.35) is just an approximation, (b) the parameters used in creating Table 5.2 are very rough (in particular, the quark mass is about right for the c, but way off for the b, and the strong coupling is only taken to one significant digit). All things considered, the agreement is surprisingly good.

Problem 5.12

For the pseudoscalar mesons, $\mathbf{S}_1 \cdot \mathbf{S}_2 = -\frac{3}{4}\hbar^2$, so

$$M = m_1 + m_2 + \frac{4m_u^2}{\hbar^2}(159)\left(-\frac{3}{4}\hbar^2\right)\frac{1}{m_1 m_2} = m_1 + m_2 - 477\frac{m_u^2}{m_1 m_2}.$$

$$\begin{aligned}
\pi &: m_1 = m_2 = m_u \Rightarrow M = 308 + 308 - 477 = \boxed{139} \\
K &: m_1 = m_u, m_2 = m_s \Rightarrow M = 308 + 483 - 477\frac{308}{483} = \boxed{487} \\
u\bar{u} &: \text{same as } \pi : 139, \quad d\bar{d} : \text{same as } \pi : 139 \\
s\bar{s} &: m_1 = m_2 = m_s \Rightarrow M = 483 + 483 - 477\left(\frac{308}{483}\right)^2 = 772 \\
\eta &: M = \frac{1}{6}(139) + \frac{1}{6}(139) + \frac{2}{3}(772) = \boxed{561} \\
\eta' &: M = \frac{1}{3}(139) + \frac{1}{3}(139) + \frac{1}{3}(772) = \boxed{350}(?!)
\end{aligned}$$

For the vector mesons, $\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{4}\hbar^2$, so

$$M = m_1 + m_2 + \frac{4m_u^2}{\hbar^2}(159)\left(\frac{1}{4}\hbar^2\right)\frac{1}{m_1 m_2} = m_1 + m_2 + 159\frac{m_u^2}{m_1 m_2}.$$

$$\begin{aligned}
\rho &: m_1 = m_2 = m_u \Rightarrow M = 308 + 308 + 159 = \boxed{775} \\
K^* &: m_1 = m_u, m_2 = m_s \Rightarrow M = 308 + 483 + 159\frac{308}{483} = \boxed{892} \\
\phi &: m_1 = m_2 = m_s \Rightarrow M = 483 + 483 + 159\left(\frac{308}{483}\right)^2 = \boxed{1031} \\
\omega &: m_1 = m_2 = m_u \Rightarrow M = 308 + 308 + 159 = \boxed{775}
\end{aligned}$$

Problem 5.13

(a) For the pseudoscalar mesons (see Problem 5.12):

$$M = m_1 + m_2 - 477\frac{m_u^2}{m_1 m_2}.$$

$$\begin{aligned}
\eta_c &: m_1 = m_2 = m_c \Rightarrow M = 1250 + 1250 - 477\left(\frac{308}{1250}\right)^2 = \boxed{2471} \\
D^0 &: m_1 = m_c,\ m_2 = m_u \Rightarrow M = 1250 + 308 - 477\frac{308}{1250} = \boxed{1440} \\
D_s^+ &: m_1 = m_c,\ m_2 = m_s \Rightarrow M = 1250 + 483 - 477\frac{(308)^2}{(1250)(483)} = \boxed{1658}
\end{aligned}$$

The observed masses are $\boxed{2980, 1865, \text{and } 1968}$, respectively.

For the vector mesons (Problem 5.12):

$$M = m_1 + m_2 + 159\frac{m_u^2}{m_1 m_2}.$$

$$\psi : m_1 = m_2 = m_c \Rightarrow M = 1250 + 1250 + 159\left(\frac{308}{1250}\right)^2 = \boxed{2510}$$

$$D^{*0} : m_1 = m_c,\ m_2 = m_u \Rightarrow M = 1250 + 308 + 159\frac{308}{1250} = \boxed{2539}$$

$$D_s^{*+} : m_1 = m_c, m_2 = m_s \Rightarrow M = 1250 + 483 + 159\frac{(308^2}{(1250)(483)} = \boxed{1758}$$

The observed masses are $\boxed{\text{3097, 2007, and 2112}}$, respectively.
(b) For the pseudoscalar bottom mesons, we have

$$u\bar{b} : m_1 = m_u,\ m_2 = m_b \Rightarrow M = 308 + 4500 - 477\frac{308}{4500} = \boxed{4775}$$

$$s\bar{b} : m_1 = m_s,\ m_2 = m_b \Rightarrow M = 483 + 4500 - 477\frac{(308)^2}{(438)(4500)} = \boxed{4960}$$

$$c\bar{b} : m_1 = m_c,\ m_2 = m_b \Rightarrow M = 1250 + 4500 - 477\frac{(308)^2}{(1250)(4500)} = \boxed{5742}$$

$$b\bar{b} : m_1 = m_2 = m_b \Rightarrow M = 4500 + 4500 - 477\left(\frac{308}{4500}\right)^2 = \boxed{8998}$$

The observed masses are $\boxed{\text{5279, 5368, 6286, ???}}$, respectively.

For the vector bottom mesons:

$$u\bar{b} : m_1 = m_u,\ m_2 = m_b \Rightarrow M = 308 + 4500 + 159\left(\frac{308}{4500}\right)^2 = \boxed{4809}$$

$$s\bar{b} : m_1 = m_s,\ m_2 = m_b \Rightarrow M = 483 + 4500 + 159\frac{(308)^2}{(483)(4500)} = \boxed{4990}$$

$$c\bar{b} : m_1 = m_c,\ m_2 = m_b \Rightarrow M = 1250 + 4500 + 159\frac{(308)^2}{(1250)(4500)} = \boxed{5753}$$

$$b\bar{b} : m_1 = m_2 = m_b \Rightarrow M = 4500 + 4500159\left(\frac{308}{4500}\right)^2 = \boxed{9001}$$

The observed masses are $\boxed{\text{???, ???, ???, and 9460}}$, respectively.

Evidently the effective masses of the heavy quarks are somewhat larger than their "bare" masses as listed in the *Particle Physics Booklet*—hardly a surprise, since the same is true of the light quarks.

Problem 5.14

The proton is composed of u, u, and d. To make it antisymmetric in $1 \leftrightarrow 2$, with appropriate normalization, we want $\boxed{p = \frac{1}{\sqrt{2}}(ud - du)u}$. The other five corners are constructed in the same way:

$$\boxed{n = \frac{1}{\sqrt{2}}(ud - du)d}, \quad \boxed{\Sigma^+ = \frac{1}{\sqrt{2}}(us - su)u}, \quad \boxed{\Sigma^- = \frac{1}{\sqrt{2}}(ds - sd)d},$$
$$\boxed{\Xi^0 = \frac{1}{\sqrt{2}}(us - su)s}, \quad \boxed{\Xi^- = \frac{1}{\sqrt{2}}(ds - sd)s}.$$

Actually, the overall *phase* of p is arbitrary, but having chosen it, the others are *not* entirely arbitrary. In terms of isospin, $p = |\frac{1}{2}\,\frac{1}{2}\rangle$; applying the isospin lowering operator (see Problem 4.23(a) for the angular momentum analog):

$$I_- p = I_- |\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(-\tfrac{1}{2})}\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = n.$$

For three particles,

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 \quad \Rightarrow \quad I_- = I_{1-} + I_{2-} + I_{3-}$$

(where the subscript indicates the particle acted upon). Meanwhile, for the quarks $I_- u = d$ and $I_- d = 0$, so

$$\begin{aligned}
I_- p &= \frac{1}{\sqrt{2}}\left[I_{1-} + I_{2-} + I_{3-}\right](udu - duu) \\
&= \frac{1}{\sqrt{2}}\left[(I_- u)du + u(I_- d)u + ud(I_- u) - (I_- d)uu - d(I_- u)u - du(I_- u)\right] \\
&= \frac{1}{\sqrt{2}}\left[ddu + 0 + udd - 0 - ddu - dud\right] \\
&= \frac{1}{\sqrt{2}}(udd - dud) = n. \quad \checkmark
\end{aligned}$$

Evidently these phases are consistent.

Now $\Sigma^+ = |1\,1\rangle$, and (Eq. 4.77)

$$I_- \Sigma^+ = I_- |1\,1\rangle = \sqrt{1(2) - 1(0)}\,|1\,0\rangle = \sqrt{2}|1\,0\rangle = \sqrt{2}\Sigma^0.$$

So (noting that s is an isosinglet, so $I_- s = 0$):

$$\begin{aligned}
I_- \Sigma^+ &= \frac{1}{\sqrt{2}}\left[I_{1-} + I_{2-} + I_{3-}\right](usu - suu) \\
&= \frac{1}{\sqrt{2}}\left[(I_- u)su + u(I_- s)u + us(I_- u) - (I_- s)uu - s(I_- u)u - su(I_- u)\right] \\
&= \frac{1}{\sqrt{2}}\left[dsu + 0 + usd - 0 - sdu - sud\right] \\
&= \frac{1}{\sqrt{2}}(dsu + usd - sdu - sud) = \sqrt{2}\Sigma^0 \Rightarrow \boxed{\Sigma^0 = \frac{1}{2}\left[(us - su)d + (ds - sd)u\right]}
\end{aligned}$$

Finally, Λ is constructed from u, d, and s; the most general linear combination of the various permutations that is antisymmetric in $1 \leftrightarrow 2$ is

$$\Lambda = \alpha(ud - du)s + \beta(us - su)d + \gamma(ds - sd)u.$$

Orthogonalize with respect to Σ^0:

$$\frac{1}{2}\left[(us - su)d + (ds - sd)u\right] \cdot \left[\alpha(ud - du)s + \beta(us - su)d + \gamma(ds - sd)u\right]$$
$$= \frac{1}{2}\left[\alpha(0) + \beta(2) + \gamma(2)\right] = 0 \quad \Rightarrow \quad \gamma = -\beta.$$

Orthogonalize with respect to ψ_A:

$$\frac{1}{\sqrt{6}}\left[uds - usd + dsu - dus + sud - sdu\right]$$
$$\cdot \left[\alpha(ud - du)s + \beta(us - su)d + \gamma(ds - sd)u\right]$$
$$= \frac{1}{\sqrt{6}}\left[\alpha(2) + \beta(-2) + \gamma(2)\right] = 0 \quad \Rightarrow \quad \alpha = \beta - \gamma = 2\beta.$$

So

$$\Lambda = \beta\left[2(ud - du)s + (us - su)d - (ds - sd)u\right].$$

Normalize:

$$\beta^2\left[2(ud - du)s + (us - su)d - (ds - sd)u\right]$$
$$\cdot \left[2(ud - du)s + (us - su)d - (ds - sd)u\right]$$
$$= \beta^2\left[4(2) + 2 + 2\right] = 12\beta^2 = 1 \quad \Rightarrow \quad \beta = \frac{1}{\sqrt{12}}.$$

$$\boxed{\Lambda = \frac{1}{\sqrt{12}}\left[2(ud - du)s + (us - su)d - (ds - sd)u\right]}.$$

Problem 5.15

We want a *color singlet* made from q and $\bar{q}$ (see Eq. 5.46). This is the color analog to the *flavor* singlet (Eq. 5.42):

$$\boxed{\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})}$$

Problem 5.16

Equation 5.62 says

$$\psi = \frac{\sqrt{2}}{3}\left[\psi_{12}(s)\psi_{12}(f) + \psi_{23}(s)\psi_{23}(f) + \psi_{13}(s)\psi_{13}(f)\right],$$

so

$$\begin{aligned}\langle\psi|\psi\rangle = \frac{2}{9}\Big[&\langle\psi_{12}(s)|\psi_{12}(s)\rangle\langle\psi_{12}(f)|\psi_{12}(f)\rangle + \langle\psi_{12}(s)|\psi_{23}(s)\rangle\langle\psi_{12}(f)|\psi_{23}(f)\rangle \\ &+\langle\psi_{12}(s)|\psi_{13}(s)\rangle\langle\psi_{12}(f)|\psi_{13}(f)\rangle + \langle\psi_{23}(s)|\psi_{12}(s)\rangle\langle\psi_{23}(f)|\psi_{12}(f)\rangle \\ &+\langle\psi_{23}(s)|\psi_{23}(s)\rangle\langle\psi_{23}(f)|\psi_{23}(f)\rangle + \langle\psi_{23}(s)|\psi_{13}(s)\rangle\langle\psi_{23}(f)|\psi_{13}(f)\rangle \\ &+\langle\psi_{13}(s)|\psi_{12}(s)\rangle\langle\psi_{13}(f)|\psi_{12}(f)\rangle + \langle\psi_{13}(s)|\psi_{23}(s)\rangle\langle\psi_{13}(f)|\psi_{23}(f)\rangle \\ &+\langle\psi_{13}(s)|\psi_{13}(s)\rangle\langle\psi_{13}(f)|\psi_{13}(f)\rangle\Big].\end{aligned}$$

These states are normalized ($\langle\psi_{12}(s)|\psi_{12}(s)\rangle = 1$, etc.), but they are not orthogonal. From Eqs. 5.51, 5.52, and 5.53, using the "typical" state $\left(\frac{1}{2}\,\frac{1}{2}\right)$:

$$\begin{aligned}\langle\psi_{12}(s)|\psi_{23}(s)\rangle &= \tfrac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\cdot(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow) = -\tfrac{1}{2} = \langle\psi_{23}(s)|\psi_{12}(s)\rangle \\ \langle\psi_{12}(s)|\psi_{13}(s)\rangle &= \tfrac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\cdot(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow) = \tfrac{1}{2} = \langle\psi_{13}(s)|\psi_{12}(s)\rangle \\ \langle\psi_{13}(s)|\psi_{23}(s)\rangle &= \tfrac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)\cdot(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow) = \tfrac{1}{2} = \langle\psi_{23}(s)|\psi_{13}(s)\rangle.\end{aligned}$$

From the figures on pp. 185-186, using the "typical" state in the upper right corner:

$$\begin{aligned}\langle\psi_{12}(f)|\psi_{23}(f)\rangle &= \tfrac{1}{2}(udu - duu)\cdot(uud - udu) = -\tfrac{1}{2} = \langle\psi_{23}(f)|\psi_{12}(f)\rangle \\ \langle\psi_{12}(f)|\psi_{13}(f)\rangle &= \tfrac{1}{2}(udu - duu)\cdot(uud - duu) = \tfrac{1}{2} = \langle\psi_{13}(f)|\psi_{12}(f)\rangle \\ \langle\psi_{13}(f)|\psi_{23}(f)\rangle &= \tfrac{1}{2}(uud - duu)\cdot(uud - udu) = \tfrac{1}{2} = \langle\psi_{23}(f)|\psi_{13}(f)\rangle.\end{aligned}$$

$$\begin{aligned}\langle\psi|\psi\rangle = \frac{2}{9}\Big[&1 + \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + 1 + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \\ &+ \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + 1\Big] = \frac{2}{9}\left[3 + \frac{6}{4}\right] = \frac{2}{9}\left[\frac{9}{2}\right] = 1. \quad \checkmark\end{aligned}$$

Problem 5.17

From Eq. 5.62, using Eqs. 5.51-53 and the diagrams on pp. 185-186, I find:

$$
\begin{aligned}
|\Sigma^+ : \ \tfrac{1}{2}\ \tfrac{1}{2}\rangle &= \frac{\sqrt{2}}{3}\left[\frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\frac{1}{\sqrt{2}}(usu - suu)\right. \\
&\quad \left. + \frac{1}{2}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(uus - usu) + \frac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(uus - suu)\right] \\
&= \frac{1}{3\sqrt{2}}\,[usu(2\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow) \\
&\quad + suu(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) + uus(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{3\sqrt{2}}\Big[&2u(\uparrow)s(\downarrow)u(\uparrow) - u(\downarrow)s(\uparrow)u(\uparrow) - u(\uparrow)s(\uparrow)u(\downarrow) \\
&+ 2s(\downarrow)u(\uparrow)u(\uparrow) - s(\uparrow)u(\downarrow)u(\uparrow) - s(\uparrow)u(\uparrow)u(\downarrow) \\
&+ 2u(\uparrow)u(\uparrow)s(\downarrow) - u(\uparrow)u(\downarrow)s(\uparrow) - u(\downarrow)u(\uparrow)s(\uparrow)\Big].
\end{aligned}
$$

$$
\begin{aligned}
&|\Lambda : \ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle \\
&= \frac{\sqrt{2}}{3}\left[\frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)\frac{1}{\sqrt{12}}(2uds - 2dus + usd - sud - dsu + sdu)\right. \\
&\quad + \frac{1}{\sqrt{2}}(\downarrow\uparrow\downarrow - \downarrow\downarrow\uparrow)\frac{1}{\sqrt{12}}(2sud - 2sdu + dus - dsu - uds + usd) \\
&\quad \left. + \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow)\frac{1}{\sqrt{12}}(2usd - 2dsu + uds - sdu - dus + sud)\right] \\
&= \frac{1}{6\sqrt{3}}\,[uds(2\uparrow\downarrow\downarrow - 2\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow + \cancel{\downarrow\downarrow\uparrow} + \uparrow\downarrow\downarrow - \cancel{\downarrow\downarrow\uparrow}) \\
&\quad + dus(-2\uparrow\downarrow\downarrow + 2\downarrow\uparrow\downarrow + \downarrow\uparrow\downarrow - \cancel{\downarrow\downarrow\uparrow} - \uparrow\downarrow\downarrow + \cancel{\downarrow\downarrow\uparrow}) \\
&\quad + usd(\uparrow\downarrow\downarrow - \cancel{\downarrow\uparrow\downarrow} + \cancel{\downarrow\uparrow\downarrow} - \downarrow\downarrow\uparrow + 2\uparrow\downarrow\downarrow - 2\downarrow\downarrow\uparrow) \\
&\quad + sud(-\cancel{\uparrow\downarrow\downarrow} + \downarrow\uparrow\downarrow + 2\downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow + \cancel{\uparrow\downarrow\downarrow} - \downarrow\downarrow\uparrow) \\
&\quad + dsu(-\uparrow\downarrow\downarrow + \cancel{\downarrow\uparrow\downarrow} - \cancel{\downarrow\uparrow\downarrow} + \downarrow\downarrow\uparrow - 2\uparrow\downarrow\downarrow + 2\downarrow\downarrow\uparrow) \\
&\quad + sdu(\cancel{\uparrow\downarrow\downarrow} - \downarrow\uparrow\downarrow - 2\downarrow\uparrow\downarrow + 2\downarrow\downarrow\uparrow - \cancel{\uparrow\downarrow\downarrow} + \downarrow\downarrow\uparrow)] \\
&= \frac{1}{2\sqrt{3}}\,\{uds(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) + dus(\downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow) + usd(\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow) \\
&\quad + sud(\downarrow\uparrow\downarrow - \downarrow\downarrow\uparrow) + dsu(\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow) + sdu(\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow)\}
\end{aligned}
$$

$$= \frac{1}{2\sqrt{3}}\Big[u(\uparrow)d(\downarrow)s(\downarrow) - u(\downarrow)d(\uparrow)s(\downarrow) + d(\downarrow)u(\uparrow)s(\downarrow) - d(\uparrow)u(\downarrow)s(\downarrow)$$
$$+ u(\uparrow)s(\downarrow)d(\downarrow) - u(\downarrow)s(\downarrow)d(\uparrow) + s(\downarrow)u(\uparrow)d(\downarrow) - s(\downarrow)u(\downarrow)d(\uparrow)$$
$$+ d(\downarrow)s(\downarrow)u(\uparrow) - d(\uparrow)s(\downarrow)u(\downarrow) + s(\downarrow)d(\downarrow)u(\uparrow) - s(\downarrow)d(\uparrow)u(\downarrow)\Big].$$

Problem 5.18

For notational simplicity, use $\psi_{ij} = -\psi_{ji}$ for the spin-1/2 functions (Eqs. 5.51, 5.52, and 5.53) and $\phi_{ij} = -\phi_{ji}$ for the octet flavor states (pp. 185-186). The following structure is antisymmetric, as you can easily check:

$$\psi = A\left[\psi_{12}\left(\phi_{31} + \phi_{32}\right) + \psi_{23}\left(\phi_{12} + \phi_{13}\right) + \psi_{31}(s)\left(\phi_{23} + \phi_{21}\right)\right]$$

Problem 5.19

(a) First construct the wave functions, using Eq. 5.62 (with 5.51, 5.52, and 5.53 for spin and the figures on pp. 185-186 for flavor):

$$|p : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}\Big[\frac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(udu - duu)$$
$$+ \frac{1}{2}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(uud - udu) + \frac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(uud - duu)\Big]$$
$$= \frac{1}{3\sqrt{2}}\left[udu(2\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow) + \text{perms}\right]$$
$$= \frac{1}{3\sqrt{2}}\left[2u(\uparrow)d(\downarrow)u(\uparrow) - u(\downarrow)d(\uparrow)u(\uparrow) - u(\uparrow)d(\uparrow)u(\downarrow) + \text{perms}\right].$$

$$|n : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}\Big[\frac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(udd - dud)$$
$$+ \frac{1}{2}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(dud - ddu) + \frac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(udd - ddu)\Big]$$
$$= \frac{1}{3\sqrt{2}}\left[udd(-2\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow) + \text{perms}\right]$$
$$= \frac{1}{3\sqrt{2}}\left[-2u(\downarrow)d(\uparrow)d(\uparrow) + u(\uparrow)d(\downarrow)d(\uparrow) + u(\uparrow)d(\uparrow)d(\downarrow) + \text{perms}\right].$$

$$|\Sigma^+ : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}\left[\frac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(usu - suu)\right.$$
$$\left. + \frac{1}{2}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(uus - usu) + \frac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(uus - suu)\right]$$
$$= \frac{1}{3\sqrt{2}}\left[usu(2\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow) + \text{perms}\right]$$
$$= \frac{1}{3\sqrt{2}}\left[2u(\uparrow)s(\downarrow)u(\uparrow) - u(\downarrow)s(\uparrow)u(\uparrow) - u(\uparrow)s(\uparrow)u(\downarrow) + \text{perms}\right].$$

$$|\Sigma^- : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}\left[\frac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(dsd - sdd)\right.$$
$$\left. + \frac{1}{2}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(dds - dsd) + \frac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(dds - sdd)\right]$$
$$= \frac{1}{3\sqrt{2}}\left[dsd(2\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow) + \text{perms}\right]$$
$$= \frac{1}{3\sqrt{2}}\left[2d(\uparrow)s(\downarrow)d(\uparrow) - d(\downarrow)s(\uparrow)d(\uparrow) - d(\uparrow)s(\uparrow)d(\downarrow) + \text{perms}\right].$$

$$|\Xi^0 : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}\left[\frac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(uss - sus)\right.$$
$$\left. + \frac{1}{2}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(sus - ssu) + \frac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(uss - ssu)\right]$$
$$= \frac{1}{3\sqrt{2}}\left[uss(-2\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow) + \text{perms}\right]$$
$$= \frac{1}{3\sqrt{2}}\left[-2u(\downarrow)s(\uparrow)s(\uparrow) + u(\uparrow)s(\downarrow)s(\uparrow) + u(\uparrow)s(\uparrow)s(\downarrow) + \text{perms}\right].$$

$$|\Xi^- : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}\left[\frac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(dss - sds)\right.$$
$$\left. + \frac{1}{2}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(sds - ssd) + \frac{1}{2}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(dss - ssd)\right]$$
$$= \frac{1}{3\sqrt{2}}\left[dss(-2\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow) + \text{perms}\right]$$
$$= \frac{1}{3\sqrt{2}}\left[-2d(\downarrow)s(\uparrow)s(\uparrow) + d(\uparrow)s(\downarrow)s(\uparrow) + d(\uparrow)s(\uparrow)s(\downarrow) + \text{perms}\right].$$

In each of these cases "perm" stands for three permutations (two in addition to the one listed), in which the odd quark occupies the first, second, or third position; Σ^0 and Λ are a little trickier, because they involve all three quarks, and there are *six* permutations:

$$
\begin{aligned}
|\Sigma^0 : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}&\left[\frac{1}{2\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(usd - sud + dsu - sdu)\right.\\
&+ \frac{1}{2\sqrt{2}}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(dus - dsu + uds - usd)\\
&\left.+ \frac{1}{2\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(uds - sdu + dus - sud)\right]\\
= \frac{1}{6}&\left[usd(2\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow) + (6\text{ perms})\right]\\
= \frac{1}{6}&\left[2u(\uparrow)s(\downarrow)d(\uparrow) - u(\downarrow)s(\uparrow)d(\uparrow) - u(\uparrow)s(\uparrow)d(\downarrow) + (6\text{ perms})\right].
\end{aligned}
$$

$$
\begin{aligned}
|\Lambda : \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}&\left[\frac{1}{2\sqrt{6}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(2uds - 2dus + usd - sud - dsu + sdu)\right.\\
&+ \frac{1}{2\sqrt{6}}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)(2sud - 2sdu + dus - dsu - uds + usd)\\
&\left.+ \frac{1}{2\sqrt{6}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(2usd - 2dsu + uds - sdu - dus + sud)\right]\\
= \frac{1}{6\sqrt{3}}&\left[3uds(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + (6\text{ perms})\right]\\
= \frac{1}{2\sqrt{3}}&\left[u(\uparrow)d(\downarrow)s(\uparrow) - u(\downarrow)d(\uparrow)s(\uparrow) + (6\text{ perms})\right].
\end{aligned}
$$

Now use Eq. 5.67 to determine the magnetic moment of the proton, as in Example 5.3:

$$
\begin{aligned}
|a\rangle &\equiv \frac{2}{3\sqrt{2}} u(\uparrow)d(\downarrow)u(\uparrow) \\
&\sum(\mu_i S_{iz})|a\rangle = \frac{2}{3\sqrt{2}}\left(\mu_u\frac{\hbar}{2} - \mu_d\frac{\hbar}{2} + \mu_u\frac{\hbar}{2}\right)|a\rangle = \frac{\hbar}{2}\frac{2}{3\sqrt{2}}(2\mu_u - \mu_d)|a\rangle \\
&\mu_a = \frac{2}{\hbar}\langle a|\sum(\mu_i S_{iz})|a\rangle = \frac{2}{9}(2\mu_u - \mu_d); \\
|b\rangle &\equiv \frac{-1}{3\sqrt{2}} u(\downarrow)d(\uparrow)u(\uparrow) \\
&\sum(\mu_i S_{iz})|b\rangle = \frac{-1}{3\sqrt{2}}\left(-\mu_u\frac{\hbar}{2} + \mu_d\frac{\hbar}{2} + \mu_u\frac{\hbar}{2}\right)|b\rangle = -\frac{\hbar}{2}\frac{1}{3\sqrt{2}}\mu_d|b\rangle \\
&\mu_b = \frac{2}{\hbar}\langle b|\sum(\mu_i S_{iz})|b\rangle = \frac{1}{18}\mu_d; \\
|c\rangle &\equiv \frac{-1}{3\sqrt{2}} u(\uparrow)d(\uparrow)u(\downarrow) \\
&\sum(\mu_i S_{iz})|c\rangle = \frac{-1}{3\sqrt{2}}\left(\mu_u\frac{\hbar}{2} + \mu_d\frac{\hbar}{2} - \mu_u\frac{\hbar}{2}\right)|c\rangle = -\frac{\hbar}{2}\frac{1}{3\sqrt{2}}\mu_d|c\rangle \\
&\mu_c = \frac{2}{\hbar}\langle c|\sum(\mu_i S_{iz})|c\rangle = \frac{1}{18}\mu_d.
\end{aligned}
$$

The total is the sum (times three for the three permutations):

$$
\mu = 3(\mu_a + \mu_b + \mu_c) = 3\left[\frac{2}{9}(2\mu_u - \mu_d) + \frac{2}{18}\mu_d\right] = \boxed{\tfrac{1}{3}(4\mu_u - \mu_d)}.
$$

Comparing the wave functions, we see that no new calculation is required for n, Σ^+, Σ^-, Ξ^0, and Ξ^- (for some of them there is an overall minus sign in the wave function, but this squares out; sometimes the permutation listed is different, but this doesn't affect the answer).

$$
\begin{aligned}
n:&\quad \text{(same as } p\text{, only with } u \leftrightarrow d\text{)}: \quad \mu = \boxed{\tfrac{1}{3}(4\mu_d - \mu_u)}. \\
\Sigma^+:&\quad \text{(same as } p\text{, only with } d \rightarrow s\text{)}: \quad \mu = \boxed{\tfrac{1}{3}(4\mu_u - \mu_s)}. \\
\Sigma^-:&\quad \text{(same as } n\text{, only with } u \rightarrow s\text{)}: \quad \mu = \boxed{\tfrac{1}{3}(4\mu_d - \mu_s)}. \\
\Xi^0:&\quad \text{(same as } n\text{, only with } d \rightarrow s\text{)}: \quad \mu = \boxed{\tfrac{1}{3}(4\mu_s - \mu_u)}. \\
\Xi^-:&\quad \text{(same as } \Xi^0\text{, only with } u \rightarrow d\text{)}: \quad \mu = \boxed{\tfrac{1}{3}(4\mu_s - \mu_d)}.
\end{aligned}
$$

As for Σ^0,

$$
\begin{aligned}
|a\rangle &\equiv \frac{1}{3}u(\uparrow)s(\downarrow)d(\uparrow)\\
&\sum(\mu_i S_{iz})|a\rangle = \frac{1}{3}\left(\mu_u\frac{\hbar}{2} - \mu_s\frac{\hbar}{2} + \mu_d\frac{\hbar}{2}\right)|a\rangle = \frac{\hbar}{2}\frac{1}{3}(\mu_u - \mu_s + \mu_d)|a\rangle\\
&\mu_a = \frac{2}{\hbar}\langle a|\sum(\mu_i S_{iz})|a\rangle = \frac{1}{9}(\mu_u - \mu_s + \mu_d);\\
|b\rangle &\equiv \frac{-1}{6}u(\downarrow)s(\uparrow)d(\uparrow)\\
&\sum(\mu_i S_{iz})|b\rangle = \frac{-1}{6}\left(-\mu_u\frac{\hbar}{2} + \mu_s\frac{\hbar}{2} + \mu_d\frac{\hbar}{2}\right)|b\rangle = -\frac{\hbar}{2}\frac{1}{6}(-\mu_u + \mu_s + \mu_d)|b\rangle\\
&\mu_b = \frac{2}{\hbar}\langle b|\sum(\mu_i S_{iz})|b\rangle = \frac{1}{36}(-\mu_u + \mu_s + \mu_d);\\
|c\rangle &\equiv \frac{-1}{6}u(\uparrow)s(\uparrow)d(\downarrow)\\
&\sum(\mu_i S_{iz})|c\rangle = \frac{-1}{6}\left(\mu_u\frac{\hbar}{2} + \mu_s\frac{\hbar}{2} - \mu_d\frac{\hbar}{2}\right)|c\rangle = -\frac{\hbar}{2}\frac{1}{6}(\mu_u + \mu_s - \mu_d)|c\rangle\\
&\mu_c = \frac{2}{\hbar}\langle c|\sum(\mu_i S_{iz})|c\rangle = \frac{1}{36}(\mu_u + \mu_s - \mu_d).
\end{aligned}
$$

The total is the sum (times six for the six permutations):

$$
\begin{aligned}
\mu &= 6(\mu_a + \mu_b + \mu_c)\\
&= 6\left[\frac{1}{9}(\mu_u - \mu_s + \mu_d) + \frac{1}{36}(-\mu_u + \mu_s + \mu_d) + \frac{1}{36}(\mu_u + \mu_s - \mu_d)\right]\\
&= \boxed{\tfrac{1}{3}(2\mu_u + 2\mu_d - \mu_s}.
\end{aligned}
$$

Finally, for Λ,

$$
\begin{aligned}
|a\rangle &\equiv \frac{1}{2\sqrt{3}}u(\uparrow)d(\downarrow)s(\uparrow)\\
&\sum(\mu_i S_{iz})|a\rangle = \frac{1}{2\sqrt{3}}\left(\mu_u\frac{\hbar}{2} - \mu_d\frac{\hbar}{2} + \mu_s\frac{\hbar}{2}\right)|a\rangle = \frac{\hbar}{2}\frac{1}{2\sqrt{3}}(\mu_u - \mu_s + \mu_s)|a\rangle\\
&\mu_a = \frac{2}{\hbar}\langle a|\sum(\mu_i S_{iz})|a\rangle = \frac{1}{12}(\mu_u - \mu_d + \mu_s);\\
|b\rangle &\equiv \frac{-1}{2\sqrt{3}}u(\downarrow)d(\uparrow)s(\uparrow)\\
&\sum(\mu_i S_{iz})|b\rangle = \frac{-1}{2\sqrt{3}}\left(-\mu_u\frac{\hbar}{2} + \mu_d\frac{\hbar}{2} + \mu_s\frac{\hbar}{2}\right)|b\rangle = -\frac{\hbar}{2}\frac{1}{2\sqrt{3}}(-\mu_u + \mu_d + \mu_s)|b\rangle\\
&\mu_b = \frac{2}{\hbar}\langle b|\sum(\mu_i S_{iz})|b\rangle = \frac{1}{12}(-\mu_u + \mu_d + \mu_s).
\end{aligned}
$$

The total is the sum (times six for the six permutations):

$$\begin{aligned}\mu &= 6(\mu_a + \mu_b \\ &= 6\left[\frac{1}{12}(\mu_u - \mu_d + \mu_s) + \frac{1}{12}(-\mu_u + \mu_d + \mu_s)\right] \\ &= \boxed{\mu_s}.\end{aligned}$$

(b) In units of the nuclear magneton, $e\hbar/2m_pc$,

$$\mu_u = \frac{2}{3}\frac{m_p}{m_u}, \quad \mu_d = -\frac{1}{3}\frac{m_p}{m_d}, \quad \mu_s = -\frac{1}{3}\frac{m_p}{m_s}$$

(Eq. 5.66), so, using

$$m_u = m_d = 336, \quad m_s = 538, \quad \text{, and } m_p = 938,$$

the magnetic moments should be

$$\begin{aligned}
p:&\quad \frac{1}{3}\left[4\left(\frac{2}{3}\right)\frac{m_p}{m_u} - \left(-\frac{1}{3}\right)\frac{m_p}{m_u}\right] = \frac{m_p}{m_u} = \frac{938}{336} = \boxed{2.79} \\
n:&\quad \frac{1}{3}\left[4\left(-\frac{1}{3}\right)\frac{m_p}{m_u} - \left(\frac{2}{3}\right)\frac{m_p}{m_u}\right] = -\left(\frac{2}{3}\right)\frac{m_p}{m_u} = -\left(\frac{2}{3}\right)\frac{938}{336} = \boxed{\text{-1.86}} \\
\Lambda:&\quad -\left(\frac{1}{3}\right)\frac{m_p}{m_s} = -\left(\frac{1}{3}\right)\frac{938}{538} = \boxed{\text{-0.58}} \\
\Sigma^+:&\quad \frac{1}{3}\left[4\left(\frac{2}{3}\right)\frac{m_p}{m_u} - \left(-\frac{1}{3}\right)\frac{m_p}{m_s}\right] = \frac{938}{9}\left(\frac{8}{336} + \frac{1}{538}\right) = \boxed{2.68} \\
\Sigma^0:&\quad \frac{1}{3}\left\{2\left[\left(\frac{2}{3}\right)\frac{m_p}{m_u} + \left(-\frac{1}{3}\right)\frac{m_p}{m_u}\right] - \left(-\frac{1}{3}\right)\frac{m_p}{m_s}\right\} = \frac{m_p}{9}\left(\frac{2}{m_u} + \frac{1}{m_s}\right) \\
&= \frac{938}{9}\left(\frac{2}{336} + \frac{1}{538}\right) = \boxed{0.81} \\
\Sigma^-:&\quad \frac{1}{3}\left[4\left(-\frac{1}{3}\right)\frac{m_p}{m_u} - \left(-\frac{1}{3}\right)\frac{m_p}{m_s}\right] = \frac{938}{9}\left(-\frac{4}{336} + \frac{1}{538}\right) = \boxed{\text{-1.05}} \\
\Xi^0:&\quad \frac{1}{3}\left[4\left(-\frac{1}{3}\right)\frac{m_p}{m_s} - \left(\frac{2}{3}\right)\frac{m_p}{m_u}\right] = \frac{938}{9}\left(-\frac{4}{538} - \frac{2}{336}\right) = \boxed{\text{-1.40}} \\
\Xi^-:&\quad \frac{1}{3}\left[4\left(-\frac{1}{3}\right)\frac{m_p}{m_s} - \left(-\frac{1}{3}\right)\frac{m_p}{m_u}\right] = \frac{938}{9}\left(-\frac{4}{538} + \frac{1}{336}\right) = \boxed{\text{-0.46}}
\end{aligned}$$

Problem 5.20

Following Problem 5.19, but with the wave function from Problem 5.18:

$$\begin{aligned}
|p:\tfrac{1}{2}\,\tfrac{1}{2}\rangle &= \frac{A}{2}\,[(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)(duu-uud+udu-uud)\\
&\quad+(\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow)(udu-duu+uud-duu)\\
&\quad-\;(\uparrow\uparrow\downarrow-\downarrow\uparrow\uparrow)(uud-udu+duu-udu)]\\
&= \frac{A}{2}\,[duu(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow-2\uparrow\uparrow\downarrow+2\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow-\uparrow\uparrow\downarrow)+\text{perms}]\\
&= \frac{3}{2}A\,[d(\uparrow)u(\downarrow)u(\uparrow)-d(\uparrow)u(\uparrow)u(\downarrow)+\text{perms}]\,.
\end{aligned}$$

$$\begin{aligned}
|n:\tfrac{1}{2}\,\tfrac{1}{2}\rangle &= \frac{A}{2}\,[(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)(ddu-udd+ddu-dud)\\
&\quad+(\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow)(udd-dud+udd-ddu)\\
&\quad-\;(\uparrow\uparrow\downarrow-\downarrow\uparrow\uparrow)(dud-ddu+dud-udd)]\\
&= \frac{A}{2}\,[udd(-\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow+2\uparrow\uparrow\downarrow-2\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow+\uparrow\uparrow\downarrow)+\text{perms}]\\
&= \frac{3}{2}A\,[u(\uparrow)d(\uparrow)d(\downarrow)-u(\uparrow)d(\downarrow)d(\uparrow)+\text{perms}]\,.
\end{aligned}$$

For the proton,

$$\begin{aligned}
|a\rangle &\equiv \frac{3A}{2}d(\uparrow)u(\downarrow)u(\uparrow)\\
&\quad \sum(\mu_i S_{iz})|a\rangle = \frac{3A}{2}\left(\mu_d\frac{\hbar}{2}-\mu_u\frac{\hbar}{2}+\mu_u\frac{\hbar}{2}\right)|a\rangle = \frac{\hbar}{2}\frac{3A}{2}(\mu_d)|a\rangle\\
&\quad \mu_a = \frac{2}{\hbar}\langle a|\sum(\mu_i S_{iz})|a\rangle = \left(\frac{3A}{2}\right)^2\mu_d;\\
|b\rangle &\equiv -\frac{3A}{2}d(\uparrow)u(\uparrow)u(\downarrow)\\
&\quad \sum(\mu_i S_{iz})|b\rangle = -\frac{3A}{2}\left(\mu_d\frac{\hbar}{2}+\mu_u\frac{\hbar}{2}-\mu_u\frac{\hbar}{2}\right)|b\rangle = -\frac{\hbar}{2}\frac{3A}{2}\mu_d|b\rangle\\
&\quad \mu_b = \frac{2}{\hbar}\langle b|\sum(\mu_i S_{iz})|b\rangle = \left(\frac{3A}{2}\right)^2\mu_d.
\end{aligned}$$

The total is the sum (times three for the three permutations):

$$\mu_p = 3(\mu_a+\mu_b) = 6\left(\frac{3A}{2}\right)^2\mu_d$$

(negative, since μ_d is). Comparing the wave functions, the neutron calculation is identical (except for an overall minus sign, which squares away), only with

$u \leftrightarrow d$. So

$$\mu_n = 6\left(\frac{3A}{2}\right)^2 \mu_u.$$

Evidently

$$\frac{\mu_n}{\mu_p} = \frac{\mu_u}{\mu_d} = \left(\frac{2}{3}\frac{m_p}{m_u}\right) / \left(-\frac{1}{3}\frac{m_p}{m_d}\right) = \boxed{-2\frac{m_d}{m_u} = -2}$$

(assuming, in the final step, that $m_u \approx m_d$, in the spirit of Eq. 5.68). No: this is totally inconsistent with the experimental value (-0.68).

Problem 5.21

The flavor states are $\rho^+ = -u\bar{d}$, $\rho^- = d\bar{u}$ (Eq. 5.38), and the $m_s = 1$ spin state is $\uparrow\uparrow$ (Eq. 4.15), so

$$|\rho^+ : 1\,1\rangle = -u(\uparrow)\bar{d}(\uparrow), \quad |\rho^- : 1\,1\rangle = d(\uparrow)\bar{u}(\uparrow).$$

Following the method of Section 5.6.2 (Eq. 5.67), and noting that in this case $S_z = \hbar/2$ for both quarks:

$$\mu_M = \frac{2}{\hbar}\langle M\,1|(\mu_1 S_{1z} + \mu_2 S_{2z})|M\,1\rangle = \langle M\,1|(\mu_1 + \mu_2)|M\,1\rangle$$

Thus

$$\mu_{\rho^+} = (\mu_u - \mu_d)\,, \quad \mu_{\rho^-} = (\mu_d - \mu_u) = -\mu_{\rho^+} \quad \checkmark$$

(tne magnetic moment of an antiquark is opposite to that of the quark). Now, to the extent that $m_u = m_d$, Eq. 5.66 says $\mu_u = -2\mu_d$, so $\mu_{\rho^+} = -2\mu_d - \mu_d = -3\mu_d$. Meanwhile, for the proton (Example 5.3)

$$\mu_p = \tfrac{1}{3}\left(4\mu_u - \mu_d\right) = \tfrac{1}{3}\left(-8\mu_d - \mu_d\right) = -3\mu_d = \mu_{\rho^+}. \quad \checkmark$$

Problem 5.22

In this case $m_1 = m_2 = m_s,\ m_3 = m_u$, so

$$M = 2m_s + m_u + A'\left[\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_s^2} + \frac{(\mathbf{S}_1 + \mathbf{S}_2)\cdot \mathbf{S}_u}{m_s m_u}\right].$$

Now $\mathbf{J} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_u$, so

$$J^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 + 2(\mathbf{S}_1 + \mathbf{S}_2)\cdot \mathbf{S}_u + S_u^2,$$

but $J^2 = S_u^2 = (3/4)\hbar^2$ (since both the quark and the Ξ have spin-1/2), so

$$(\mathbf{S}_1 + \mathbf{S}_2) \cdot \mathbf{S}_u = -\tfrac{1}{2}(\mathbf{S}_1 + \mathbf{S}_2)^2.$$

The flavor wave function is symmetric in $1 \leftrightarrow 2$, so the spin function must also be symmetric—the two s quarks are in the triplet (spin 1) state:

$$(\mathbf{S}_1 + \mathbf{S}_2)^2 = 2\hbar^2, \quad (\mathbf{S}_1 + \mathbf{S}_2) \cdot \mathbf{S}_u = -\hbar^2, \quad \mathbf{S}_1 \cdot \mathbf{S}_2 = \tfrac{1}{4}\hbar^2.$$

$$M = 2m_s + m_u + A'\left[\frac{\hbar^2}{4m_s^2} - \frac{\hbar^2}{m_s m_u}\right] = m_u + 2m_s + \frac{\hbar^2}{4}A'\left(\frac{1}{m_s^2} - \frac{4}{m_u m_s}\right).$$

This confirms Eq. 5.83. Numerically, using $m_u = 363$, $m_s = 538$, $A' = 50(2m_u/\hbar)^2$:

$$M = 2(538) + 363 + 50(363)^2\left[\frac{1}{(538)^2} - \frac{4}{(538)(363)}\right] = \boxed{1327\ \text{MeV}/c^2}.$$

(The observed value is 1315.)

Problem 5.23

For hydrogen (Problem 5.2)

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}}e^{-r/a} \quad \Rightarrow \quad |\psi_{100}(0)|^2 = \frac{1}{\pi a^3}.$$

For positronium the Bohr radius is doubled (Eq. 5.28): $|\psi_{100}(0)|^2 = 1/[\pi(2a)^3]$. Now $a = \hbar^2/me^2$ (Eq. 5.13) and $\alpha = e^2/\hbar c$ (Eq. 5.11), so $a = \hbar/\alpha mc$, and hence

$$|\psi_{100}(0)|^2 = \boxed{\frac{1}{\pi}\left(\frac{\alpha mc}{2\hbar}\right)^3}.$$

6
The Feynman Calculus

Problem 6.1

Number there at time t: $N(t) = N_0 e^{-\Gamma t}$.
Number still there at time $t + dt$:

$$N(t+dt) = N_0 e^{-\Gamma(t+dt)} = N_0 e^{-\Gamma t} e^{-\Gamma dt} \cong N_0 e^{-\Gamma t}(1 - \Gamma\, dt).$$

Number that decay between t and $t + dt$:

$$N(t) - N(t+dt) \cong N_0 e^{-\Gamma t}(1 - 1 + \Gamma\, dt) = N_0 e^{-\Gamma t}\Gamma\, dt.$$

Fraction that decay between t and $t + dt$: $\Gamma e^{-\Gamma t} dt$.
Probability that an individual selected at random from the initial sample will decay between t and $t + dt$:

$$p(t)dt = \Gamma e^{-\Gamma t} dt.$$

Mean lifetime:

$$\tau = \int_0^\infty t p(t)\, dt = \Gamma \int_0^\infty t e^{-\Gamma t}\, dt = \Gamma\left(\frac{1}{\Gamma^2}\right) = \boxed{\frac{1}{\Gamma}.}$$

[Note the distinction between the probability that a given particle will decay in the next instant dt (which is $\Gamma\, dt$) and the probability that an individual in the initial sample will decay between t and $t + dt$ (which is $\Gamma e^{-\Gamma t} dt$); the difference is that in the latter case there are fewer and fewer *around* to decay (hence the factor $e^{-\Gamma t}$).]

Problem 6.2

$$N(t_{1/2}) = \frac{1}{2}N_0 = N_0 e^{-\Gamma t_{1/2}} \Rightarrow e^{\Gamma t_{1/2}} = 2 \Rightarrow \Gamma t_{1/2} = \ln 2 \Rightarrow \boxed{t_{1/2} = \tau \ln 2.}$$

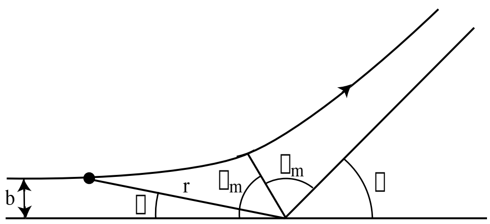
Problem 6.3

(a) $N = 10^6 e^{-(2.2\times10^{-5})/(2.2\times10^{-6})} = 10^6 e^{-10} = \left(4.58 \times 10^{-5}\right) \times 10^6 = \boxed{46}$ (approximately).

(b)

$$e^{-1/(2.6\times10^{-8})} = e^{-3.85\times10^7} = \left(10^{\log_{10} e}\right)^{(-3.85\times10^7)} = \left(10^{0.434}\right)^{(-3.85\times10^7)}$$
$$= 10^{-1.67\times10^7} = \boxed{10^{-16,700,000}} \quad \text{(pretty small!)}$$

Problem 6.4



Use polar coordinates (r, ϕ): $\mathbf{v} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}$.

Energy is conserved:

$$E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r).$$

Angular momentum is conserved:

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = mr\,\hat{r} \times (\dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}) = mr^2\dot{\phi}(\hat{r} \times \hat{\phi}) = mr^2\dot{\phi}\,\hat{z}, \text{ or } L = mr^2\dot{\phi}.$$

In the distant past, $E = \frac{1}{2}mv_0^2$ and $L = bmv_0$, so $L = bm\sqrt{2E/m}$. So conservation of angular momentum says

$$\dot{\phi} = \frac{L}{mr^2} = \frac{bm\sqrt{2E/m}}{mr^2} = \frac{b}{r^2}\sqrt{\frac{2E}{m}},$$

and conservation of energy becomes

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{b^2}{r^4}\frac{2E}{m} + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{Eb^2}{r^2} + V(r), \text{ or}$$

$$\dot{r}^2 = \frac{2}{m}\left[E - \frac{Eb^2}{r^2} - V(r)\right].$$

However we're not interested in r as a function of t, but rather r as a function of ϕ:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\dot{\phi} = \frac{dr}{d\phi}\frac{b}{r^2}\sqrt{\frac{2E}{m}}.$$

It pays to change variables, letting $u \equiv 1/r$, so that

$$\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}.$$

$$\dot{r} = -r^2\frac{du}{d\phi}\frac{b}{r^2}\sqrt{\frac{2E}{m}} = -b\sqrt{\frac{2E}{m}}\frac{du}{d\phi}. \quad \text{Thus}$$

$$b^2\frac{2E}{m}\left(\frac{du}{d\phi}\right)^2 = \frac{2E}{m}\left(1 - \frac{b^2}{r^2} - \frac{V}{E}\right) \Longrightarrow \left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2} - \frac{V}{b^2E}, \text{ or}$$

$$\frac{du}{d\phi} = \frac{1}{b}\sqrt{1 - \frac{b^2}{r^2} - \frac{V}{E}} = \frac{1}{b}\sqrt{1 - b^2u^2 - \frac{V}{E}}; \qquad \frac{b\,du}{\sqrt{1 - b^2u^2 - V/E}} = d\phi.$$

Integrate in from distant past ($\phi = 0$, $r = \infty \Longrightarrow u = 0$) to point of closest approach ($\phi = \phi_m$, $r = r_{\min} \Rightarrow u = u_{\max}$, the point at which $du/d\phi = 0$, which is to say, where the radical goes to zero):

$$\boxed{\phi_m = b\int_0^{u_m}\frac{du}{\sqrt{1 - b^2u^2 - V/E}}}; \quad \text{then } \boxed{\theta = \pi - 2\phi_m} \quad \text{(see diagram).}$$

(a) So far, this is true for *any* potential. Now we put in $V(r) = \dfrac{k}{r^2} = ku^2$:

$$\phi_m = b\int_0^{u_m}\frac{du}{\sqrt{1 - b^2u^2 - \frac{k}{E}u^2}} = \frac{b}{\sqrt{b^2 + k/E}}\int_0^{u_m}\frac{du}{\sqrt{\frac{1}{b^2+k/E} - u^2}}$$
$$= \frac{1}{\sqrt{1 + \frac{k}{b^2E}}}\int_0^{u_m}\frac{du}{\sqrt{u_m^2 - u^2}},$$

where $u_m^2 = \frac{1}{b^2+k/E}$.

$$\phi_m = \frac{1}{\sqrt{1 + \frac{k}{b^2E}}}\sin^{-1}\left(\frac{u}{u_m}\right)\Bigg|_0^{u_m} = \frac{1}{\sqrt{1 + \frac{k}{b^2E}}}\sin^{-1}(1) = \frac{\pi}{2}\frac{1}{\sqrt{1 + \frac{k}{b^2E}}}$$

$$\theta = \pi - 2\phi_m = \pi - \pi\frac{1}{\sqrt{1 + \frac{k}{b^2E}}} = \boxed{\pi\left(1 - \frac{1}{\sqrt{1 + (k/b^2E)}}\right)}.$$

(b) According to Eq. 6.10, $\dfrac{d\sigma}{d\Omega} = \left|\dfrac{b}{\sin\theta}\dfrac{db}{d\theta}\right|$. In this case,

$$\frac{d\theta}{db} = \pi\frac{1}{2}\frac{1}{[1+(k/b^2E)]^{3/2}}\left(\frac{k}{E}\right)\left(-\frac{2}{b^3}\right) = -\frac{k\pi}{E}\left(b^2+\frac{k}{E}\right)^{-3/2}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{E}{k\pi}\left(b^2+\frac{k}{E}\right)^{3/2}.\quad \text{But we want it in terms of } \theta:$$

$$1-\frac{\theta}{\pi} = \frac{1}{\sqrt{1+(k/b^2E)}} \Longrightarrow 1+\frac{k}{b^2E} = \frac{1}{(1-\frac{\theta}{\pi})^2}$$

$$\Longrightarrow \frac{k}{b^2E} = \frac{1}{(1-\frac{\theta}{\pi})^2} - 1 = \frac{1-1+2\frac{\theta}{\pi}-\frac{\theta^2}{\pi^2}}{(1-\frac{\theta}{\pi})^2} = \frac{\theta(2\pi-\theta)}{(\pi-\theta)^2}$$

$$\frac{b^2E}{k} = \frac{(\pi-\theta)^2}{\theta(2\pi-\theta)};\quad b^2 = \frac{k}{E}\frac{(\pi-\theta)^2}{\theta(2\pi-\theta)}$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{b^4}{\sin\theta}\frac{E}{k\pi}\left(1+\frac{k}{b^2E}\right)^{3/2} = \frac{1}{\sin\theta}\frac{k^2}{E^2}\frac{(\pi-\theta)^4}{\theta^2(2\pi-\theta)^2}\frac{E}{k\pi}\left(\frac{\pi}{\pi-\theta}\right)^3$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\pi^2}{\sin\theta}\frac{k}{E}\frac{(\pi-\theta)}{\theta^2(2\pi-\theta)^2}}$$

(c)

$$\sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \frac{\pi^2k}{E}\int\frac{1}{\sin\theta}\frac{(\pi-\theta)}{\theta^2(2\pi-\theta)^2}(\sin\theta\, d\theta\, d\phi)$$

$$= \frac{2\pi^3k}{E}\int_0^\pi\frac{(\pi-\theta)}{\theta^2(2\pi-\theta)^2}d\theta = \boxed{\text{Infinity.}}$$

Problem 6.5

$$m_1c = \sqrt{r^2+m_2^2c^2}+\sqrt{r^2+m_3^2c^2}.\quad \text{Square:}$$

$$m_1^2c^2 = r^2+m_2^2c^2+r^2+m_3^2c^2+2\sqrt{r^2+m_2^2c^2}\sqrt{r^2+m_3^2c^2}$$

$$\frac{c^2}{2}(m_1^2-m_2^2-m_3^2)-r^2 = \sqrt{r^2+m_2^2c^2}\sqrt{r^2+m_3^2c^2}.\quad \text{Square again:}$$

$$\frac{c^4}{4}(m_1^2-m_2^2-m_3^2)^2-r^2c^2(m_1^2-m_2^2-m_3^2)+\cancel{r^4} = \cancel{r^4}+r^2c^2(m_2^2+m_3^2)+m_2^2m_3^2c^4$$

$$\frac{c^4}{4}\left[(m_1^2-m_2^2-m_3^2)^2-4m_2^2m_3^2\right] = r^2c^2(m_2^2+m_3^2+m_1^2-m_2^2-m_3^2) = r^2m_1^2c^2$$

$$r^2 = \frac{c^2}{4m_1^2}\left[m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 + 2m_2^2m_3^2 - 4m_2^2m_3^2\right]$$

$$\boxed{r = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}} \quad \checkmark$$

Problem 6.6

Plug $\mathcal{M} = \alpha m_\pi c$ into Eq. 6.35, with $S = 1/2$:

$$\Gamma = \frac{|\mathbf{p}|}{16\pi\hbar m_\pi^2 c}(\alpha m_\pi c)^2 = \frac{\alpha^2 c}{16\pi\hbar}|\mathbf{p}|.$$

But $E_\gamma = \frac{1}{2}m_\pi c^2 \Rightarrow |\mathbf{p}_\gamma| = E_\gamma/c = \frac{1}{2}m_\pi c.$ So $\Gamma = \alpha^2 m_\pi c^2/32\pi\hbar.$

$$\tau = \frac{1}{\Gamma} = \frac{32\pi\hbar}{\alpha^2 m_\pi c^2} = \frac{32\pi(6.58\times10^{-22})(137)^2}{135} = \boxed{9.2\times10^{-18}\,\text{s.}}$$

The experimental value is 8.4×10^{-17} s, so this estimate is off by a factor of 10.

Problem 6.7

(a) In the *CM* frame, $\mathbf{p}_2 = -\mathbf{p}_1$, so $p_1 = \left(\frac{E_1}{c}, \mathbf{p}_1\right)$, $p_2 = \left(\frac{E_2}{c}, -\mathbf{p}_1\right)$.

$$p_1 \cdot p_2 = \frac{E_1}{c}\frac{E_2}{c} - \mathbf{p}_1 \cdot (-\mathbf{p}_1) = \frac{E_1E_2}{c^2} + \mathbf{p}_1^2$$

$$(p_1\cdot p_2)^2 - (m_1m_2c^2)^2 = \left(\frac{E_1E_2}{c^2} + \mathbf{p}_1^2\right)^2 - (m_1m_2c^2)^2$$
$$= \frac{E_1^2E_2^2}{c^4} + 2\frac{E_1E_2}{c^2}\mathbf{p}_1^2 + \mathbf{p}_1^4 - m_1^2m_2^2c^4$$

But $m_1^2c^2 = \frac{E_1^2}{c^2} - \mathbf{p}_1^2$, and $m_2^2c^2 = \frac{E_2^2}{c^2} - \mathbf{p}_2^2 = \frac{E_2^2}{c^2} - \mathbf{p}_1^2$. So:

$$(p_1\cdot p_2)^2 - (m_1m_2c^2)^2$$
$$= \cancel{\frac{E_1^2E_2^2}{c^4}} + 2\frac{E_1E_2}{c^2}\mathbf{p}_1^2 + \cancel{\mathbf{p}_1^4} - \underbrace{\left(\frac{E_1^2}{c^2} - \mathbf{p}_1^2\right)\left(\frac{E_2^2}{c^2} - \mathbf{p}_1^2\right)}_{\cancel{\frac{E_1^2E_2^2}{c^4}} - \mathbf{p}_1^2\frac{(E_1^2+E_2^2)}{c^2} + \cancel{\mathbf{p}_1^4}}$$
$$= \frac{1}{c^2}\mathbf{p}_1^2(E_1^2 + E_2^2 + 2E_1E_2) = \frac{1}{c^2}\mathbf{p}_1^2(E_1+E_2)^2.$$

$$\therefore \boxed{\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = \frac{1}{c}|\mathbf{p}_1|(E_1 + E_2).}$$

(b) In the lab frame, $p_1 = \left(\frac{E_1}{c}, \mathbf{p}_1\right)$, $p_2 = (m_2 c, \mathbf{0})$, so $p_1 \cdot p_2 = E_1 m_2$.

$$\therefore (p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2 = E_1^2 m_2^2 - m_1^2 m_2^2 c^4 = m_2^2(E_1^2 - m_1^2 c^4) = m_2^2(\mathbf{p}_1^2 c^2).$$

$$\therefore \boxed{\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = |\mathbf{p}_1| m_2 c.}$$

Problem 6.8

Start with Eq. 6.47:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}.$$

This is for the CM frame, but here the target particle is so heavy that it barely moves, and hence the lab and the CM are effectively the same. Particle 1 bounces off particle 2; its energy is unchanged, and hence so too is the magnitude of its momentum: $|\mathbf{p}_f| = |\mathbf{p}_i|$, and $(E_1 + E_2) = (E_1 + m_2 c^2) \approx m_2 c^2$, so

$$\frac{d\sigma}{d\Omega} = \boxed{\left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(m_b c^2)^2}}.$$

($S = 1$, since 1 and 2 are different particles).

Problem 6.9

Using the result of Problem 6.7 (b) in Eq. 6.38:

$$d\sigma = \frac{\hbar^2 S}{4|\mathbf{p}_1| m_2 c} \left(\frac{1}{4\pi}\right)^2 \int |\mathcal{M}|^2 \, \delta\left(\frac{E_1 + E_2}{c} - |\mathbf{p}_3| - |\mathbf{p}_4|\right)$$
$$\times \; \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \frac{d^3\mathbf{p}_3 \, d^3\mathbf{p}_4}{|\mathbf{p}_3| \, |\mathbf{p}_4|}.$$

Here $\mathbf{p}_2 = 0$ and $E_2 = m_2 c^2$. Doing the $\mathbf{p}_4$ integral:

$$d\sigma = \frac{\hbar^2 S}{64\pi^2 |\mathbf{p}_1| m_2 c} \int |\mathcal{M}|^2 \delta\left(\frac{E_1}{c} + m_2 c - |\mathbf{p}_3| - |\mathbf{p}_1 - \mathbf{p}_3|\right) \frac{d^3\mathbf{p}_3}{|\mathbf{p}_3||\mathbf{p}_1 - \mathbf{p}_3|}.$$

Now $d^3\mathbf{p}_3 = |\mathbf{p}_3|^2\, d|\mathbf{p}_3|\, d\Omega$, and $(\mathbf{p}_1 - \mathbf{p}_3)^2 = |\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta$ (where θ is the scattering angle for particle 3). So

$$\frac{d\sigma}{d\Omega} = \frac{\hbar^2 S}{64\pi^2|\mathbf{p}_1|m_2 c}\int_0^\infty |\mathcal{M}|^2 \frac{|\mathbf{p}_3|^2\, d|\mathbf{p}_3|}{|\mathbf{p}_3|\sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta}}$$
$$\times\, \delta\left(\frac{E_1}{c} + m_2 c - |\mathbf{p_3}| - \sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta}\right)$$

Let $z \equiv |\mathbf{p_3}| + \sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta}$; then

$$\begin{aligned}
\frac{dz}{d|\mathbf{p_3}|} &= 1 + \frac{1}{2}\frac{2|\mathbf{p_3}| - 2|\mathbf{p_1}|\cos\theta}{\sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta}} \\
&= \frac{\sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta} + |\mathbf{p}_3| - |\mathbf{p}_1|\cos\theta}{\sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta}} \\
&= \frac{z - |\mathbf{p}_1|\cos\theta}{\sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta}}.
\end{aligned}$$

$$\frac{d|\mathbf{p}_3|}{\sqrt{|\mathbf{p_1}|^2 + |\mathbf{p_3}|^2 - 2|\mathbf{p_1}||\mathbf{p_3}|\cos\theta}} = \frac{dz}{z - |\mathbf{p}_1|\cos\theta}.$$ Therefore:

$$\frac{d\sigma}{d\Omega} = \frac{\hbar^2 S}{64\pi^2|\mathbf{p}_1|m_2 c}\int |\mathcal{M}|^2 \frac{|\mathbf{p_3}|\, dz}{(z - |\mathbf{p_1}|\cos\theta)}\delta\left(\frac{E_1}{c} + m_2 c - z\right)$$

[Here $|\mathbf{p_3}|$ is a function of z.]

$$= \frac{\hbar^2 S}{64\pi^2|\mathbf{p}_1|m_2 c}|\mathcal{M}|^2 \frac{|\mathbf{p_3}|}{\frac{E_1}{c} + m_2 c - |\mathbf{p_1}|\cos\theta}$$

[Here $|\mathbf{p_3}|$ is the value when $z = \dfrac{E_1}{c} + m_2 c$.]

$$= \boxed{\left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2|\mathbf{p_3}|}{m_2|\mathbf{p_1}|(E_1 + m_2 c^2 - |\mathbf{p_1}|c\cos\theta)}}$$

Problem 6.10

(a) Insert the result from Problem 6.7(b) into Eq. 6.38 (and note that $\mathbf{p}_2 = 0$):

$$d\sigma = \frac{\hbar^2 S}{4|\mathbf{p}_1|m_2 c}\frac{1}{(4\pi)^2}\int |\mathcal{M}|^2 \delta^4(p_1 + p_2 - p_3 - p_4)\frac{d^3\mathbf{p}_3\, d^3\mathbf{p}_4}{\sqrt{\mathbf{p}_3^2 + m_1^2c^2}\sqrt{\mathbf{p}_4^2 + m_2^2c^2}}$$
$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{|\mathbf{p}_1|cm_2}\int |\mathcal{M}|^2 \delta\left(\frac{E_1 + E_2}{c} - \sqrt{\mathbf{p}_3^2 + m_1^2c^2} - \sqrt{\mathbf{p}_4^2 + m_2^2c^2}\right)$$
$$\times\, \delta^3(\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{p}_4)\frac{d^3\mathbf{p}_3\, d^3\mathbf{p}_4}{\sqrt{\mathbf{p}_3^2 + m_1^2c^2}\sqrt{\mathbf{p}_4^2 + m_2^2c^2}}.$$

Doing the $\mathbf{p}_4$ integral, and setting $d^3\mathbf{p}_3 = |\mathbf{p}_3|^2\, d|\mathbf{p}_3|\, d\Omega$:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{|\mathbf{p}_1|cm_2}\int |\mathcal{M}|^2$$
$$\times\, \frac{\delta\left(\frac{E_1}{c} + m_2 c - \sqrt{\mathbf{p}_3^2 + m_1^2c^2} - \sqrt{(\mathbf{p}_1 - \mathbf{p}_3)^2 + m_2^2c^2}\right)}{\sqrt{\mathbf{p}_3^2 + m_1^2c^2}\,\sqrt{(\mathbf{p}_1 - \mathbf{p}_3)^2 + m_2^2c^2}}|\mathbf{p}_3|^2 d\mathbf{p}_3.$$

Now

$$(\mathbf{p}_1 - \mathbf{p}_3)^2 = \mathbf{p}_1^2 + \mathbf{p}_3^2 - 2\mathbf{p}_1\cdot\mathbf{p}_3 = \mathbf{p}_1^2 + \mathbf{p}_3^2 - 2|\mathbf{p}_1|\,|\mathbf{p}_3|\,\cos\theta,$$

where θ is the scattering angle. Let $r \equiv |\mathbf{p}_3|$, for simplicity:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{|\mathbf{p}_1|cm_2}\int |\mathcal{M}|^2$$
$$\times\, \frac{\delta\left(\frac{E_1}{c} + m_2 c - \sqrt{r^2 + m_1^2c^2} - \sqrt{r^2 - 2r|\mathbf{p}_1|\cos\theta + \mathbf{p}_1^2 + m_2^2c^2}\right)}{\sqrt{r^2 + m_1^2c^2}\,\sqrt{r^2 - 2r|\mathbf{p}_1|\cos\theta + \mathbf{p}_1^2 + m_2^2c^2}}r^2\, dr.$$

Let $z \equiv \sqrt{r^2 + m_1^2c^2} + \sqrt{r^2 - 2r|\mathbf{p}_1|\cos\theta + \mathbf{p}_1^2 + m_2^2c^2}$. Then

$$\frac{dz}{dr} = \frac{r}{\sqrt{r^2 + m_1^2c^2}} + \frac{r - |\mathbf{p}_1|\cos\theta}{\sqrt{r^2 - 2r|\mathbf{p}_1|\cos\theta + \mathbf{p}_1^2 + m_2^2c^2}}$$
$$= \frac{rz - |\mathbf{p}_1|\cos\theta\sqrt{r^2 + m_1^2c^2}}{\sqrt{r^2 + m_1^2c^2}\,\sqrt{r^2 - 2r|\mathbf{p}_1|\cos\theta + \mathbf{p}_1^2 + m_2^2c^2}}.$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{|\mathbf{p}_1|cm_2}\int |\mathcal{M}|^2 \frac{\delta\left(\frac{E_1}{c} + m_2 c - z\right)}{\left|rz - |\mathbf{p}_1|\cos\theta\sqrt{r^2 + m_1^2c^2}\right|}r^2\, dz$$

(here r is a function, implicitly, of z)

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{|\mathbf{p}_1|cm_2}|\mathcal{M}|^2 \frac{r^2}{\left|r(E_1/c + m_2c) - |\mathbf{p}_1|\cos\theta\sqrt{r^2 + m_1^2c^2}\right|}$$

(now r is the value of $|\mathbf{p}_3|$ dictated by conservation of energy and momentum, so $\sqrt{r^2 + m_1^2c^2} = \sqrt{\mathbf{p}_3^2 + m_1^2c^2} = E_3/c$).

$$\frac{d\sigma}{d\Omega} = \boxed{\left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{|\mathbf{p}_1|m_2} \frac{\mathbf{p}_3^2}{|(E_1 + m_2c^2)|\mathbf{p}_3| - E_3|\mathbf{p}_1|\cos\theta|}}.$$

Note that E_3 and $\mathbf{p}_3$ are *functions of* θ (for a given incident energy E_1). The final result can be written in a number of different (equivalent) ways; perhaps the tidiest is

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{|\mathbf{p}_1|m_2} \frac{|\mathbf{p}_3|^3}{|E_1E_3m_2 - E_4m_1^2c^2|}.$$

(b) In this case $|\mathbf{p}_1| = E_1/c$ and $|\mathbf{p}_3| = E_3/c$, so

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2c}{E_1m_2} \frac{E_3^2/c^2}{|(E_1 + m_2c^2)E_3/c - (E_1/c)E_3\cos\theta|} \\ &= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{E_1m_2} \frac{E_3}{|E_1 + m_2c^2 - E_1\cos\theta|}.\end{aligned}$$

Now

$$p_1 + p_2 = p_3 + p_4 \Rightarrow p_1 - p_3 = p_4 - p_2$$

$$p_1^2 + p_3^2 - 2p_1 \cdot p_3 = p_4^2 + p_2^2 - 2p_2 \cdot p_4 \Rightarrow p_1 \cdot p_3 = p_2 \cdot p_4 - m_2^2c^2$$

$$\frac{E_1E_3}{c^2} - |\mathbf{p}_1||\mathbf{p}_3|\cos\theta = \frac{E_2E_4}{c^2} - \mathbf{p}_2 \cdot \mathbf{p}_4 - m_2^2c^2 = m_2E_4 - m_2^2c^2$$

$$\begin{aligned}E_1E_3(1 - \cos\theta) &= m_2c^2(E_4 - m_2c^2) = m_2c^2(E_1 + E_2 - E_3 - m_2c^2) \\ &= (E_1 - E_3)m_2c^2\end{aligned}$$

$$E_3\left[E_1(1 - \cos\theta) + m_2c^2\right] = E_1m_2c^2, \quad \frac{1}{E_1 + m_2c^2 - E_1\cos\theta} = \frac{E_3}{E_1m_2c^2}.$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{E_1m_2} \frac{E_3^2}{E_1m_2c^2} = S|\mathcal{M}|^2 \left(\frac{\hbar E_3}{8\pi m_2E_1c}\right)^2. \quad \checkmark$$

Problem 6.11

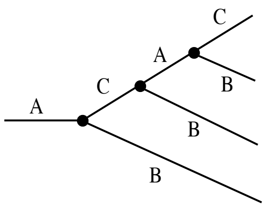
(a) No [See **(b)**.]

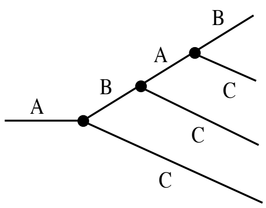
(b) Allowed if (and only if) n_A, n_B, and n_C are either *all even* or *all odd*.

Proof: *(necessary)* Take the allowed diagram and snip every internal line. We now have $n'_A = n'_B = n'_C = N$ 'external' lines, where N is the number of vertices. When we now reconnect the internal lines, each join removes two 'external' lines of one species. Thus when they're all back together we have $n_A = N - 2I_A$, $n_B = N - 2I_B$, and $n_C = N - 2I_C$, where I_A is the number of internal A lines, and so on. Clearly, they're all even, or all odd, depending on the number of vertices.
(sufficient) Given n_A, n_B, and n_C, pick the largest of them (say, n_A) and draw that number of vertices, with A, B, C as 'external' lines on each one. Now just connect up B lines in pairs (converting two 'external' lines into one internal line, each time you do so), until you're down to n_B – as long as n_A and n_B are either both even or both odd, you will obviously be able to do so. Now do the same for n_C. We have constructed a diagram, then, with n_A external A lines, n_B external B lines, and n_C external C lines.

(c) In view of **(b)** we'll need *either* 3 B's and one C or 3 C's and one B:

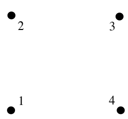
$$\boxed{A \longrightarrow B + B + B + C \quad \text{or } A \longrightarrow B + C + C + C.}$$



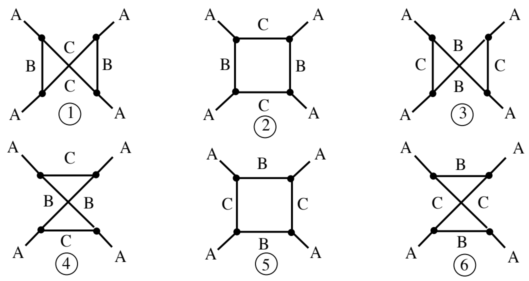


Problem 6.12

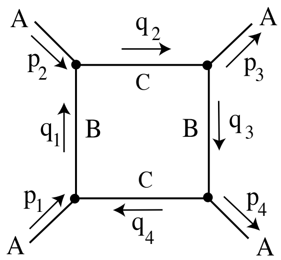
(a) Start with the four vertices:



An external A will attach to each of these (with 4-momentum labeled by the number of the vertex). The B from vertex 1 could connect to vertex 2 (in which case the C line goes to 3 or 4), or to 3 (in which case the C line goes to 2 or 4), or to 4 (in which case the C line goes to 1 or 2). So there are six diagrams in all:



(b) Let's start with diagram 2:



$$\iiiint (-ig)^4 \frac{i}{q_1^2 - m_B^2c^2}\frac{i}{q_2^2 - m_C^2c^2}\frac{i}{q_3^2 - m_B^2c^2}\frac{i}{q_4^2 - m_C^2c^2}(2\pi)^4$$
$$\times\, \delta^4(p_1 + q_4 - q_1)(2\pi)^4\delta^4(p_2 + q_1 - q_2)(2\pi)^4\delta^4(q_2 - q_3 - p_3)$$
$$\times\, (2\pi)^4\delta^4(q_3 - p_4 - q_4)\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}$$
$$= g^4 \iiiint \frac{\delta^4(p_1 + q_4 - q_1)\delta^4(p_2 + q_1 - q_2)\delta^4(q_2 - q_3 - p_3)}{q_1^2 q_2^2 q_3^2}$$
$$\times\, \frac{\delta^4(q_3 - p_4 - q_4)}{q_4^2} d^4q_1\, d^4q_2\, d^4q_3\, d^4q_4$$

Do the q_4 integral ($\Rightarrow q_4 = q_1 - p_1$) and the q_3 integral ($\Rightarrow q_3 = q_2 - p_3$):

$$= g^4 \iint \frac{\delta^4(p_2 + q_1 - q_2)\delta^4(q_2 - p_3 - p_4 - q_1 + p_1)}{q_1^2 q_2^2 (q_2 - p_3)^2 (q_1 - p_1)^2} d^4q_1\, d^4q_2$$

Do the q_2 integral ($\Rightarrow q_2 = p_2 + q_1$), and drop the subscript on q_1:

$$= g^4 \int \frac{\delta^4(p_2 + \not{q} - p_3 - p_4 - \not{q} + p_1)}{q^2(p_2+q)^2(p_2+q-p_3)^2(q-p_1)^2}\, d^4q$$

Cancel $(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)$, and multiply by i:

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q+p_2)^2(q+p_2-p_3)^2(q-p_1)^2}\, d^4q.$$

The diagram 5 is the same, only with $m_C \leftrightarrow m_B$. Since, however, we assumed $m_C = m_B = 0$, the amplitude is identical to diagram 2. For diagram 1 (and hence also 3) we switch $p_3 \leftrightarrow p_4$:

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q+p_2)^2(q+p_2-p_4)^2(q-p_1)^2}\, d^4q;$$

for diagrams 4 and 6 we switch $p_2 \leftrightarrow p_3$:

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q+p_3)^2(q+p_3-p_2)^2(q-p_1)^2}\, d^4q.$$

Thus the full amplitude is

$$\boxed{\mathcal{M} = 2i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q-p_1)^2}\left\{\frac{1}{(q+p_2)^2(q+p_2-p_3)^2} + \frac{1}{(q+p_2)^2(q+p_2-p_4)^2} + \frac{1}{(q+p_3)^2(q+p_3-p_2)^2}\right\} d^4q.}$$

(There are other ways of writing this, of course, which amount to redefining the integration variable q.)

Problem 6.13

With $m_C = 0$, Eq. 6.55 becomes

$$\mathcal{M} = g^2\left\{\frac{1}{(p_4-p_2)^2} + \frac{1}{(p_3-p_2)^2}\right\}$$

With $m_B = 0$,

$$(p_4-p_2)^2 = p_4^2 + p_2^2 - 2p_4\cdot p_2 = m_B^2c^2 + m_A^2c^2 - 2p_4\cdot p_2 = m_A^2c^2 - 2p_4\cdot p_2$$

$$(p_3-p_2)^2 = p_3^2 + p_2^2 - 2p_3\cdot p_2 = m_B^2c^2 + m_A^2c^2 - 2p_3\cdot p_2 = m_A^2c^2 - 2p_3\cdot p_2.$$

Now $E_1 = E_2$ and $E_3 = E_4$, so conservation of energy $(E_1 + E_2 = E_3 + E_4)$ says $E_1 = E_2 = E_3 = E_4 \equiv E$. Thus:

$$p_1 = \left(\frac{E}{c}, \mathbf{p}_1\right); \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}_1\right); \quad p_3 = \left(\frac{E}{c}, \mathbf{p}_3\right); \quad p_4 = \left(\frac{E}{c}, -\mathbf{p}_3\right),$$

so $p_2 \cdot p_4 = E^2/c^2 - \mathbf{p}_1 \cdot \mathbf{p}_3; \quad p_2 \cdot p_3 = E^2/c^2 + \mathbf{p}_1 \cdot \mathbf{p}_3.$

$$\begin{aligned}\mathcal{M} &= g^2\left(\frac{1}{m_A^2c^2 - 2E^2/c^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_3} + \frac{1}{m_A^2c^2 - 2E^2/c^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_3}\right)\\ &= g^2\left[\frac{2\left(m_A^2c^2 - 2E^2/c^2\right)}{\left(m_A^2c^2 - 2E^2/c^2\right)^2 - 4(\mathbf{p}_1 \cdot \mathbf{p}_3)^2}\right]\end{aligned}$$

Now $\mathbf{p}_1 \cdot \mathbf{p}_3 = |\mathbf{p}_1||\mathbf{p}_3|\cos\theta$ (Fig. 6.10), and

$$\mathbf{p}_1^2 = \frac{E^2}{c^2} - m_A^2c^2; \quad \mathbf{p}_3^2 = \frac{E^2}{c^2} - m_B^2c^2 = \frac{E^2}{c^2}.$$

So $(\mathbf{p}_1 \cdot \mathbf{p}_3)^2 = \left(E^2/c^2 - m_A^2c^2\right)\left(E^2/c^2\right)\cos^2\theta$.

$$\mathcal{M} = 2g^2\left\{\frac{\left(m_A^2c^2 - 2E^2/c^2\right)}{\left(m_A^2c^2 - 2E^2/c^2\right)^2 - 4(E^2/c^2)\left(E^2/c^2 - m_A^2c^2\right)\cos^2\theta}\right\}$$

According to Eq. 6.47, $\dfrac{d\sigma}{d\Omega} = \left(\dfrac{\hbar c}{8\pi}\right)^2 \dfrac{S|\mathcal{M}|^2}{(E_1+E_2)^2}\dfrac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$. Here $S = 1/2$,

$$(E_1 + E_2)^2 = 4E^2, \quad |\mathbf{p}_f| = E/c, \quad |\mathbf{p}_i| = \sqrt{E^2/c^2 - m_A^2c^2} \quad \text{so}$$

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left(\frac{g^2\hbar c}{8\pi}\right)^2 \frac{1}{2E^2}\frac{E/c}{\sqrt{E^2/c^2 - m_A^2c^2}}\\ &\quad\times \frac{\left(m_A^2c^2 - 2E^2/c^2\right)^2}{\left[\left(m_A^2c^2 - 2E^2/c^2\right)^2 - 4(E^2/c^2)\left(E^2/c^2 - m_A^2c^2\right)\cos^2\theta\right]^2}\end{aligned}$$

$$\boxed{= \left(\frac{\hbar g^2c^3}{8\pi}\right)^2 \frac{1}{2E\sqrt{E^2 - m_A^2c^4}}\frac{\left(m_A^2c^4 - 2E^2\right)^2}{\left[\left(m_A^2c^4 - 2E^2\right)^2 - 4E^2\left(E^2 - m_A^2c^4\right)\cos^2\theta\right]^2}}$$

The total cross-section is

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \int \frac{d\sigma}{d\Omega}\sin\theta\, d\theta\, d\phi = 2\pi A\int_0^\pi \frac{\sin\theta\, d\theta}{(a - b\cos^2\theta)^2}, \quad \text{where}$$

$$A \equiv \left(\frac{\hbar g^2 c^3}{8\pi}\right)^2 \frac{1}{2E\sqrt{E^2 - m_A^2 c^4}} \left(m_A^2 c^4 - 2E^2\right)^2;$$

$$a \equiv \left(m_A^2 c^4 - 2E^2\right)^2; \quad b \equiv 4E^2\left(E^2 - m_A^2 c^4\right).$$

$$\int_0^\pi \frac{\sin\theta\, d\theta}{(a - b\cos^2\theta)^2} = \int_{-1}^1 \frac{dx}{(a - bx^2)^2} \quad (\text{letting } x \equiv \cos\theta)$$
$$= \frac{1}{a}\left[\frac{1}{a-b} + \frac{1}{\sqrt{ab}}\tanh^{-1}\sqrt{\frac{b}{a}}\right].$$

Here

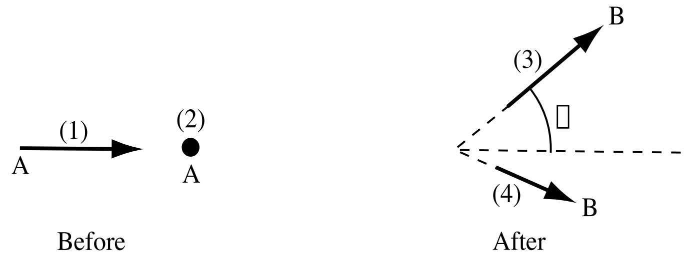
$$a - b = (m_A c^2)^4 - 4E(m_A c^2)^2 + 4E^4 - 4E^2(E^2 - m_A^2 c^4) = (m_A c^2)^4, \text{ so}$$

$$\sigma = (2\pi)\left(\frac{\hbar g^2 c^3}{8\pi}\right)^2 \frac{1}{2E\sqrt{E^2 - m_A^2 c^4}} \left\{\frac{1}{(m_A c^2)^4}\right.$$
$$\left. + \frac{1}{(2E^2 - m_A^2 c^4)\, 2E\sqrt{E^2 - m_A^2 c^4}} \tanh^{-1}\left[\frac{2E\sqrt{E^2 - m_A^2 c^4}}{(2E^2 - m_A^2 c^4)}\right]\right\}$$
$$= \frac{1}{2\pi}\left(\frac{\hbar g^2 c^3}{8}\right)^2 \frac{1}{E^2\left(E^2 - m_A^2 c^4\right)} \left[\frac{2E\sqrt{E^2 - m_A^2 c^4}}{(m_A c^2)^4}\right.$$
$$\left. + \frac{1}{(2E^2 - m_A^2 c^4)} \tanh^{-1}\left(\frac{2E\sqrt{E^2 - m_A^2 c^4}}{2E^2 - m_A^2 c^4}\right)\right].$$

Problem 6.14

Quoting Eq. 6.55, with $m_C = 0$:

$$\mathcal{M} = g^2\left[\frac{1}{(p_4 - p_2)^2} + \frac{1}{(p_3 - p_2)^2}\right].$$



Here

$$p_1 = \left(\frac{E}{c}, \mathbf{p}\right), \quad p_2 = (mc, \mathbf{0}), \quad p_3 = \frac{E_3}{c}(1, \hat{p}_3), \quad p_4 = \frac{E_4}{c}(1, \hat{p}_4).$$

$$\begin{aligned}
(p_3 - p_2)^2 &= p_3^2 + p_2^2 - 2p_2 \cdot p_3 = 0 + (mc)^2 - 2mE_3, \\
(p_4 - p_2)^2 &= p_4^2 + p_2^2 - 2p_2 \cdot p_4 = 0 + (mc)^2 - 2mE_4 \\
&= (p_1 - p_3)^2 = p_3^2 + p_1^2 - 2p_1 \cdot p_3 \\
&= 0 + (mc)^2 - 2\left(\frac{EE_3}{c^2} - \frac{E_3}{c}|\mathbf{p}|\cos\theta\right)
\end{aligned}$$

So

$$E_4 = \frac{E_3}{mc^2}\left(E - |\mathbf{p}|c\cos\theta\right)$$

But the incident proton is nonrelativistic, so $E \approx mc^2 \gg |\mathbf{p}|c$, and hence $E_4 \approx EE_3/mc^2 \approx E_3$. Meanwhile, conservation of energy says $E + mc^2 = E_3 + E_4 \approx 2E_3$, so $E_3 \approx E_4 \approx mc^2$, and therefore

$$(p_4 - p_2)^2 \approx mc^2 - 2(mc)^2 = -(mc)^2 = (p_3 - p_2)^2$$

$$\mathcal{M} = g^2\left[-\frac{1}{(mc)^2} - \frac{1}{(mc)^2}\right] = -2\left(\frac{g}{mc}\right)^2.$$

Using the result of Problem 6.9:

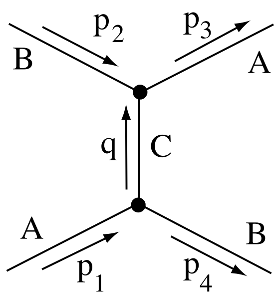
$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{\hbar}{8\pi}\right)^2 \frac{\frac{1}{2}|\mathcal{M}|^2|\mathbf{p}_3|}{m|\mathbf{p}|[E + mc^2 - |\mathbf{p}|c\cos\theta)} \quad \text{where} \quad |\mathbf{p}_3| = \frac{E_3}{c} \approx mc \\
&\approx \frac{1}{2}\left(\frac{\hbar}{8\pi}\right)^2 \frac{mc}{m|\mathbf{p}|(2mc^2)} 4\left(\frac{g}{mc}\right)^4 = \left(\frac{\hbar g^2}{8\pi}\right)^2 \frac{1}{|\mathbf{p}|(mc)^5}.
\end{aligned}$$

But $|\mathbf{p}| \approx mv$, so, in the nonrelativistic limit,

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\hbar g^2}{8\pi(mc)^3}\right)^2\left(\frac{c}{v}\right)} \quad \boxed{\sigma = 4\pi\left(\frac{\hbar g^2}{8\pi(mc)^3}\right)^2\left(\frac{c}{v}\right)}.$$

Problem 6.15

(a)



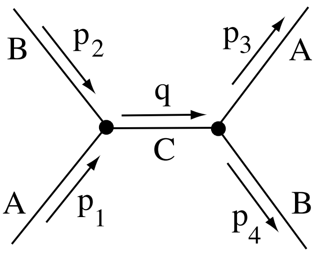


Diagram 1:

$$\int (-ig)^2 \frac{i}{q^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 - p_4 - q)(2\pi)^4 \delta^4(q + p_2 - p_3) \frac{d^4 q}{(2\pi)^4}$$

$$= \frac{-ig^2}{(p_1 - p_4)^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4), \quad \mathcal{M}_1 = \frac{g^2}{(p_1 - p_4)^2 - m_C^2 c^2}.$$

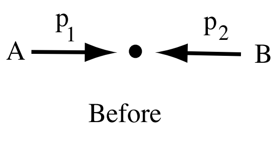
Diagram 2:

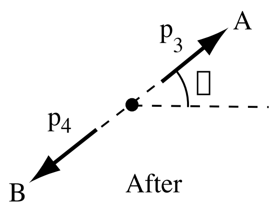
$$\int (-ig)^2 \frac{i}{q^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - q)(2\pi)^4 \delta^4(q - p_3 - p_4) \frac{d^4 q}{(2\pi)^4}$$

$$= \frac{-ig^2}{(p_1 + p_2)^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4), \quad \mathcal{M}_2 = \frac{g^2}{(p_1 + p_2)^2 - m_C^2 c^2}.$$

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \boxed{g^2 \left[\frac{1}{(p_1 - p_4)^2 - m_C^2 c^2} + \frac{1}{(p_1 + p_2)^2 - m_C^2 c^2} \right]}$$

(b)





$$p_1 = \left(\frac{E}{c}, \mathbf{p}_1\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}_1\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}_3\right), \quad p_4 = \left(\frac{E}{c}, -\mathbf{p}_3\right)$$

$$(p_1 + p_2) = \left(\frac{2E}{c}, \mathbf{0}\right), \quad \text{so} \quad (p_1 + p_2)^2 = 4\frac{E^2}{c^2}$$

$$p_1 - p_4 = (0, \mathbf{p}_1 + \mathbf{p}_3), \quad \text{so}$$

$$(p_1 - p_4)^2 = -(\mathbf{p}_1 + \mathbf{p}_3)^2 = -(\mathbf{p}_1^2 + \mathbf{p}_3^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_3)$$
$$= -(2\mathbf{p}_1^2 + 2|\mathbf{p}_1||\mathbf{p}_1|\cos\theta) = -4\mathbf{p}_1^2 \cos^2\theta/2.$$
$$\mathcal{M} = \frac{g^2}{4}\left(\frac{c^2}{E^2} - \frac{1}{\mathbf{p}_1^2\cos^2\theta/2}\right).$$

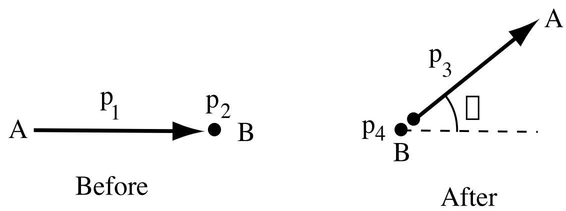
$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(2E)^2}\frac{|\mathbf{p}_4|}{|\mathbf{p}_1|}; \quad |\mathbf{p}_4| = |\mathbf{p}_1|,$$
$$= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{(2E)^2}\left(\frac{g^2}{4}\right)^2\left(\frac{c^2}{E^2} - \frac{1}{\mathbf{p}_1^2\cos^2\theta/2}\right)^2$$

But $\mathbf{p}_1^2 = E^2/c^2 - m_A^2 c^2$, so

$$\left(\frac{c^2}{E^2} - \frac{1}{\mathbf{p}_1^2\cos^2\theta/2}\right) = \frac{c^2}{E^2} - \frac{c^2}{(E^2 - m^2c^4)\cos^2\theta/2}$$
$$= c^2\frac{\left[(E^2 - m^2c^4)\cos^2\theta/2 - E^2\right]}{E^2(E^2 - m^2c^4)\cos^2\theta/2} = c^2\frac{\left[E^2\left(\cos^2\theta/2 - 1\right) - m^2c^4\cos^2\theta/2\right]}{E^2(E^2 - m^2c^4)\cos^2\theta/2}$$
$$= -c^2\frac{\left(E^2\sin^2\theta/2 + m^2c^4\cos^2\theta/2\right)}{E^2(E^2 - m^2c^4)\cos^2\theta/2} = -c^2\frac{\left(E^2\tan^2\theta/2 + m^2c^4\right)}{E^2(E^2 - m^2c^4)}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left\{\frac{g^2\hbar c^3}{64\pi}\frac{\left(E^2\tan^2\frac{\theta}{2} + m^2c^4\right)}{E^3(E^2 - m^2c^4)}\right\}^2.}$$

(c)



$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_B c}\right)^2 |\mathcal{M}|^2$$

$$\left.\begin{aligned} p_1 &= \left(\frac{E}{c}, \mathbf{p}_1\right), \quad p_2 = (m_B c, \mathbf{0}) \\ p_3 &= \left(\frac{E}{c}, \mathbf{p}_3\right), \quad p_4 = (m_B c, \mathbf{0}) \end{aligned}\right\} \quad \text{From these, we get:}$$

$$(p_1 - p_4) = \left(\frac{E}{c} - m_B c, \mathbf{p}_1\right); \quad (p_1 + p_2) = \left(\frac{E}{c} + m_B c, \mathbf{p}_1\right).$$

$$\begin{aligned}(p_1 - p_4)^2 - m_C^2 c^2 &= \left(\frac{E}{c} - m_B c\right)^2 - \mathbf{p}_1^2 - m_C^2 c^2 \\ &= \left(\frac{E}{c} - m_B c\right)^2 - \left(\frac{E^2}{c^2} - m_A^2 c^2\right) - m_C^2 c^2 \cong m_B^2 c^2. \\ (p_1 + p_2)^2 - m_C^2 c^2 &= \left(\frac{E}{c} + m_B c\right)^2 - \mathbf{p}_1^2 - m_C^2 c^2 \\ &= \left(\frac{E}{c} + m_B c\right)^2 - \left(\frac{E^2}{c^2} - m_A^2 c^2\right) - m_C^2 c^2 \cong m_B^2 c^2.\end{aligned}$$

$$\Rightarrow \mathcal{M} = g^2 \frac{2}{m_B^2 c^2}. \quad \boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\hbar g^2}{4\pi m_B^3 c^3}\right)^2.}$$

(d)

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = 4\pi \left(\frac{d\sigma}{d\Omega}\right) \Longrightarrow \boxed{\sigma = \frac{1}{\pi}\left(\frac{\hbar g^2}{2 m_B^3 c^3}\right)^2.}$$

7
Quantum Electrodynamics

Problem 7.1

From Eq. 3.8,

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3. \end{aligned}$$

Lowering a spatial index costs a minus sign, so

$$\begin{aligned} x_0' &= \gamma(x_0 + \beta x_1) \\ -x_1' &= \gamma(-x_1 - \beta x_0) \Rightarrow x_1' = \gamma(x_1 + \beta x_0) \\ x_2' &= x_2 \\ x_3' &= x_3. \end{aligned}$$

These are the transformation rules for covariant 4-vectors. Now

$$(\partial_\mu \phi)' = \frac{\partial \phi}{\partial x^{\mu\prime}} = \frac{\partial \phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial x^{\mu\prime}} = \frac{\partial x^\nu}{\partial x^{\mu\prime}}(\partial_\nu \phi),$$

but the inverse Lorentz transformations (Eq. 3.3) say

$$\begin{aligned} x^0 &= \gamma(x^{0\prime} + \beta x^{1\prime}) \\ x^1 &= \gamma(x^{1\prime} + \beta x^{0\prime}) \\ x^2 &= x^{2\prime} \\ x^3 &= x^{3\prime}, \end{aligned}$$

from which it follows that

$$\frac{\partial x^0}{\partial x^{0\prime}} = \gamma, \quad \frac{\partial x^0}{\partial x^{1\prime}} = \gamma\beta, \quad \frac{\partial x^1}{\partial x^{0\prime}} = \gamma\beta, \quad \frac{\partial x^1}{\partial x^{1\prime}} = \gamma, \quad \frac{\partial x^2}{\partial x^{2\prime}} = 1, \quad \frac{\partial x^3}{\partial x^{3\prime}} = 1,$$

and all the rest are zero. Accordingly,

$$\begin{aligned}
(\partial_0\phi)' &= (\partial_0\phi)\frac{\partial x^0}{\partial x^{0'}} + (\partial_1\phi)\frac{\partial x^1}{\partial x^{0'}} = \gamma\left[(\partial_0\phi) + \beta(\partial_1\phi)\right] \\
(\partial_1\phi)' &= (\partial_0\phi)\frac{\partial x^0}{\partial x^{1'}} + (\partial_1\phi)\frac{\partial x^1}{\partial x^{1'}} = \gamma\left[(\partial_1\phi) + \beta(\partial_0\phi)\right] \\
(\partial_2\phi)' &= (\partial_2\phi)\frac{\partial x^2}{\partial x^{2'}} = (\partial_2\phi) \\
(\partial_3\phi)' &= (\partial_3\phi)\frac{\partial x^3}{\partial x^{3'}} = (\partial_3\phi),
\end{aligned}$$

so $\partial_\mu\phi$ transforms as a covariant 4-vector (justifying the placement of the index).

Problem 7.2

$$\begin{aligned}
\{\gamma^0,\gamma^0\} &= 2(\gamma^0)^2 = 2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 2g^{00}, \\
\{\gamma^0,\gamma^i\} &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
&= \begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix} + \begin{pmatrix}0 & -\sigma^i\\ -\sigma^i & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 2g^{0i}, \\
\{\gamma^i,\gamma^j\} &= \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} \\
&= \begin{pmatrix}-\sigma^i\sigma^j & 0\\ 0 & -\sigma^i\sigma^j\end{pmatrix} + \begin{pmatrix}-\sigma^j\sigma^i & 0\\ 0 & -\sigma^j\sigma^i\end{pmatrix} = \begin{pmatrix}-\{\sigma^i,\sigma^j\} & 0\\ 0 & -\{\sigma^i,\sigma^j\}\end{pmatrix}.
\end{aligned}$$

But $\{\sigma^i,\sigma^j\} = 2\delta_{ij}$ (Problem 4.20), so

$$\{\gamma^i,\gamma^j\} = -2\delta_{ij}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 2g^{ij}.$$

Conclusion: $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ for all μ,ν. ✓

Problem 7.3

From Eq. 7.46,

$$\begin{aligned}
u^{(1)\dagger}u^{(1)} &= N^* \begin{pmatrix} 1 & 0 & \frac{cp_z}{(E+mc^2)} & \frac{c(p_x-ip_y)}{(E+mc^2)} \end{pmatrix} N \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{(E+mc^2)} \\ \frac{c(p_x+ip_y)}{(E+mc^2)} \end{pmatrix} \\
&= |N|^2 \left\{ 1 + 0 + \frac{c^2 p_z^2}{(E+mc^2)^2} + \frac{c^2(p_x - ip_y)(p_x+ip_y)}{(E+mc^2)^2} \right\} \\
&= \frac{|N|^2}{(E+mc^2)^2} \left\{ (E+mc^2)^2 + c^2 p_z^2 + c^2(p_x^2 + p_y^2) \right\} \\
&= \frac{|N|^2}{(E+mc^2)^2} \left\{ (E+mc^2)^2 + c^2 \mathbf{p}^2 \right\}.
\end{aligned}$$

But $c^2\mathbf{p}^2 = E^2 - m^2c^4 = (E - mc^2)(E + mc^2)$, so

$$u^{(1)\dagger}u^{(1)} = \frac{|N|^2}{(E+mc^2)} \{E + mc^2 + E - mc^2\} = \frac{2E|N|^2}{(E+mc^2)}.$$

Likewise, $u^{(2)\dagger}u^{(2)} = \frac{2E|N|^2}{(E+mc^2)}$ (same calculation with last two terms switched). From Eq. 7.43, this is equal to $2E/c$, so

$$\frac{2E|N|^2}{(E+mc^2)} = \frac{2E}{c}, \quad \text{or} \quad \boxed{|N| = \sqrt{\frac{E+mc^2}{c}}}.$$

Similarly, starting with Eq. 7.47,

$$\begin{aligned}
v^{(2)\dagger}v^{(2)} &= |N|^2 \begin{pmatrix} \frac{cp_z}{(E+mc^2)} & \frac{c(p_x-ip_y)}{(E+mc^2)} & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{cp_z}{(E+mc^2)} \\ \frac{c(p_x+ip_y)}{(E+mc^2)} \\ 1 \\ 0 \end{pmatrix} \\
&= |N|^2 \left\{ \frac{\mathbf{p}^2 c^2}{(E+mc^2)^2} + 1 \right\} \\
&= \frac{|N|^2}{(E+mc^2)^2} \left\{ (E+mc^2)(E-mc^2) + (E+mc^2)^2 \right\} \\
&= \frac{|N|^2}{(E+mc^2)} \left[E - mc^2 + E + mc^2 \right] = \frac{2E|N|^2}{(E+mc^2)}
\end{aligned}$$

(and the same for $v^{(1)}$). So the normalization constant is the same for all of them.

Problem 7.4

$$u^{(1)\dagger}u^{(2)} = |N|^2 \begin{pmatrix} 1 & 0 & \frac{cp_z}{(E+mc^2)} & \frac{c(p_x-ip_y)}{(E+mc^2)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{(E+mc^2)} \\ -\frac{cp_z}{(E+mc^2)} \end{pmatrix}$$
$$= |N|^2 \left\{ 0 + 0 + \frac{c^2 p_z(p_x - ip_y)}{(E+mc^2)^2} - \frac{c^2 p_z(p_x - ip_y)}{(E+mc^2)^2} \right\} = 0. \quad \checkmark$$

$$v^{(1)\dagger}v^{(2)} = -|N|^2 \begin{pmatrix} \frac{c(p_x+ip_y)}{(E+mc^2)} & \frac{-cp_z}{(E+mc^2)} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{cp_z}{(E+mc^2)} \\ \frac{c(p_x+ip_y)}{(E+mc^2)} \\ 1 \\ 0 \end{pmatrix}$$
$$= -|N|^2 \left\{ \frac{c^2 p_z(p_x + ip_y)}{(E+mc^2)^2} - \frac{c^2 p_z(p_x + ip_y)}{(E+mc^2)^2} + 0 + 0 \right\} = 0. \quad \checkmark$$

$$u^{(1)\dagger}v^{(1)} = |N|^2 \begin{pmatrix} 1 & 0 & \frac{cp_z}{(E+mc^2)} & \frac{c(p_x-ip_y)}{(E+mc^2)} \end{pmatrix} \begin{pmatrix} \frac{c(p_x-ip_y)}{(E+mc^2)} \\ \frac{-cp_z}{(E+mc^2)} \\ 0 \\ 1 \end{pmatrix}$$
$$= |N|^2 \left\{ \frac{c(p_x - ip_y)}{(E+mc^2)} + 0 + 0 + \frac{c(p_x - ip_y)}{(E+mc^2)} \right\} = 2(p_x - ip_y).$$

NO: $u^{(1)}$ and $v^{(1)}$ are *not* orthogonal.

Problem 7.5

In the nonrelativistic limit $\mathbf{p} = m\mathbf{v}$ and $E = mc^2$, so

$$\frac{cp_z}{E+mc^2} = \frac{cmv_z}{2mc^2} = \frac{v_z}{2c}, \qquad \frac{c(p_x \pm ip_y)}{E+mc^2} = \frac{cm(v_x \pm iv_y)}{2mc^2} = \frac{(v_x \pm iv_y)}{2c}.$$

So u_A is of order 1, but u_B is of order v/c. $\checkmark$

Problem 7.6

In this case $p_x = p_y = 0$, and $cp_z = c|\mathbf{p}| = \sqrt{E^2 - m^2c^4} = \sqrt{(E - mc^2)(E + mc^2)}$.

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{\sqrt{(E-mc^2)(E+mc^2)}}{(E+mc^2)} \\ 0 \end{pmatrix} = \frac{\sqrt{E+mc^2}}{\sqrt{c}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{\frac{(E-mc^2)}{(E+mc^2)}} \\ 0 \end{pmatrix} = \boxed{\frac{1}{\sqrt{c}} \begin{pmatrix} \sqrt{E+mc^2} \\ 0 \\ \sqrt{E-mc^2} \\ 0 \end{pmatrix}}.$$

$$\boxed{u^{(2)} = \frac{1}{\sqrt{c}} \begin{pmatrix} 0 \\ \sqrt{E+mc^2} \\ 0 \\ -\sqrt{E-mc^2} \end{pmatrix}}.$$

$$v^{(1)} = \frac{\sqrt{E+mc^2}}{\sqrt{c}} \begin{pmatrix} 0 \\ \frac{-\sqrt{(E-mc^2)(E+mc^2)}}{(E+mc^2)} \\ 0 \\ 1 \end{pmatrix} = \boxed{\frac{1}{\sqrt{c}} \begin{pmatrix} 0 \\ -\sqrt{E-mc^2} \\ 0 \\ \sqrt{E+mc^2} \end{pmatrix}}$$

$$\boxed{v^{(2)} = -\frac{1}{\sqrt{c}} \begin{pmatrix} \sqrt{E-mc^2} \\ 0 \\ \sqrt{E+mc^2} \\ 0 \end{pmatrix}}.$$

From Eq. 7.51,

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

so

$$S_z u^{(1)} = \frac{\hbar}{2} u^{(1)}, \; S_z u^{(2)} = -\frac{\hbar}{2} u^{(2)}, \; S_z v^{(1)} = -\frac{\hbar}{2} v^{(1)}, \; S_z v^{(2)} = \frac{\hbar}{2} v^{(2)}. \quad \checkmark$$

Eigenvalues are $\boxed{\frac{\hbar}{2}, -\frac{\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \text{respectively}}$.

Problem 7.7

Let $u^{(\pm)} \equiv a u^{(1)} + b u^{(2)}$. This automatically satisfies the Dirac equation (7.49); we need to pick the constants a and b such that $u^{(\pm)}$ is a (normalized) eigenstate of the helicity operator:

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\Sigma})\, u^{(\pm)} = \pm u^{(\pm)}.$$

Now

$$\mathbf{p}\cdot\mathbf{\Sigma} = \begin{pmatrix} \mathbf{p}\cdot\sigma & 0 \\ 0 & \mathbf{p}\cdot\sigma \end{pmatrix}, \quad \text{and} \quad \mathbf{p}\cdot\sigma = \begin{pmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{pmatrix},$$

so

$$\begin{aligned}
(\mathbf{p}\cdot\mathbf{\Sigma})\, u^{(\pm)} &= \begin{pmatrix} \mathbf{p}\cdot\sigma & 0 \\ 0 & \mathbf{p}\cdot\sigma \end{pmatrix} \left\{ aN \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{(E+mc^2)} \\ \frac{c(p_x+ip_y)}{(E+mc^2)} \end{pmatrix} + bN \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{(E+mc^2)} \\ \frac{-cp_z}{(E+mc^2)} \end{pmatrix} \right\} \\
&= N \left\{ a \begin{pmatrix} (\mathbf{p}\cdot\sigma)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{c(\mathbf{p}\cdot\boldsymbol{\sigma})}{E+mc^2} \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \end{pmatrix} + b \begin{pmatrix} (\mathbf{p}\cdot\sigma)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{c(\mathbf{p}\cdot\boldsymbol{\sigma})}{E+mc^2} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \end{pmatrix} \right\} \\
&= N \left\{ a \begin{pmatrix} p_z \\ p_x + ip_y \\ \frac{c\mathbf{p}^2}{E+mc^2} \\ 0 \end{pmatrix} + b \begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ \frac{c\mathbf{p}^2}{E+mc^2} \end{pmatrix} \right\} = \pm|\mathbf{p}|u^{(\pm)} \\
&= \pm|\mathbf{p}|N \left\{ a \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{(E+mc^2)} \\ \frac{c(p_x+ip_y)}{(E+mc^2)} \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{(E+mc^2)} \\ \frac{-cp_z}{(E+mc^2)} \end{pmatrix} \right\}.
\end{aligned}$$

Equating the top elements of the last two equations (you can check for yourself that the other three components yield the same condition),

$$ap_z + b(p_x - ip_y) = \pm|\mathbf{p}|a, \quad \text{or} \quad b = \left(\frac{\pm|\mathbf{p}| - p_z}{p_x - ip_y} \right) a.$$

So

$$\begin{aligned}
u^{(\pm)} &= Na \begin{pmatrix} 1 \\ \frac{\pm|\mathbf{p}|-p_z}{p_x-ip_y} \\ \frac{\pm c|\mathbf{p}|}{(E+mc^2)} \\ \frac{c|\mathbf{p}|(|\mathbf{p}|\mp p_z)}{(E+mc^2)(p_x-ip_y)} \end{pmatrix} = \frac{Na}{p_z \pm |\mathbf{p}|} \begin{pmatrix} p_z \pm |\mathbf{p}| \\ \frac{(|\mathbf{p}|^2-p_z^2)}{p_x-ip_y} \\ \frac{\pm c|\mathbf{p}|(p_z\pm|\mathbf{p}|)}{(E+mc^2)} \\ \frac{\pm c|\mathbf{p}|(|\mathbf{p}|^2-p_z^2)}{(E+mc^2)(p_x-ip_y)} \end{pmatrix} \\
&= A \begin{pmatrix} p_z \pm |\mathbf{p}| \\ p_x + ip_y \\ \frac{\pm c|\mathbf{p}|(p_z\pm|\mathbf{p}|)}{(E+mc^2)} \\ \frac{\pm c|\mathbf{p}|(p_x+ip_y)}{(E+mc^2)} \end{pmatrix} = A \begin{pmatrix} u \\ \frac{\pm c|\mathbf{p}|}{(E+mc^2)} u \end{pmatrix},
\end{aligned}$$

where

$$u \equiv \begin{pmatrix} p_z \pm |\mathbf{p}| \\ p_x + ip_y \end{pmatrix}$$

and $A \equiv Na/(p_z \pm |\mathbf{p}|)$ is determined by normalization:

$$\begin{aligned}
u^{(\pm)\dagger} u^{(\pm)} &= |A|^2 \begin{pmatrix} u^\dagger & \frac{\pm c|\mathbf{p}|}{E+mc^2} u^\dagger \end{pmatrix} \begin{pmatrix} u \\ \frac{\pm c|\mathbf{p}|}{E+mc^2} u \end{pmatrix} = |A|^2 \left(u^\dagger u + \frac{c^2\mathbf{p}^2}{(e+mc^2)^2} u^\dagger u \right) \\
&= |A|^2 \left(1 + \frac{E^2 - m^2c^4}{(E+mc^2)^2} \right) \left((p_z \pm |\mathbf{p}|) \;\; (p_x - ip_y) \right) \begin{pmatrix} p_z \pm |\mathbf{p}| \\ p_x + ip_y \end{pmatrix} \\
&= |A|^2 \left(1 + \frac{E - mc^2}{E + mc^2} \right) \left(p_z^2 \pm 2p_z|\mathbf{p}| + \mathbf{p}^2 + p_x^2 + p_y^2 \right) \\
&= |A|^2 \frac{2E}{(E+mc^2)} 2|\mathbf{p}| \left(|\mathbf{p}| \pm p_z \right).
\end{aligned}$$

Using the convention in Eq. 7.43, this is equal to $2E/c$, so

$$|A|^2 = \frac{(E+mc^2)}{2|\mathbf{p}|c\,(|\mathbf{p}| \pm p_z)}. \quad \checkmark$$

Problem 7.8

(a) Equation 7.19 says $\gamma^0 p^0 - \boldsymbol{\gamma} \cdot \mathbf{p} - mc = 0$. Multiply from the left by $c\gamma^0$, noting that $(\gamma^0)^2 = 1$:

$$H \equiv cp^0 = c\gamma^0(\boldsymbol{\gamma} \cdot \mathbf{p} + mc).$$

(b)

$$\begin{aligned}
[H, L_i] &= \sum_{j,k} \epsilon_{ijk} [H, r^j p^k] = \sum_{j,k,l} \epsilon_{ijk} c\gamma^0 \gamma^l [p^l, r^j] p^k = \sum_{j,k,l} \epsilon_{ijk} c\gamma^0 \gamma^l (-i\hbar\delta^{jl}) p^k \\
&= -i\hbar c\gamma^0 \sum_{j,k} \epsilon_{ijk} \gamma^j p^k = -i\hbar c\gamma^0 (\boldsymbol{\gamma} \times \mathbf{p})_i, \\
[H, \mathbf{L}] &= -i\hbar c\gamma^0 (\boldsymbol{\gamma} \times \mathbf{p}).
\end{aligned}$$

(c) Noting that γ^0 commutes with $\boldsymbol{\Sigma}$, we have

$$\begin{aligned}
[H, S_i] &= \frac{\hbar}{2} [c\gamma^0(\boldsymbol{\gamma} \cdot \mathbf{p} + mc), \Sigma_i] = \frac{c\hbar}{2} \sum_j p^j [\gamma^0 \gamma^j, \Sigma_i]; \\
\gamma^0 \gamma^j &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}; \\
[\gamma^0 \gamma^j, \Sigma_i] &= \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} - \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\sigma_j, \sigma_i] \\ [\sigma_j, \sigma_i] & 0 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
[\sigma_j, \sigma_i] &= -2i \sum_k \epsilon_{ijk} \sigma_k, \quad \text{so} \\
[\gamma^0 \gamma^j, \Sigma_i] &= -2i \sum_k \epsilon_{ijk} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = -2i\gamma^0 \sum_k \epsilon_{ijk} \gamma^k, \\
[H, S_i] &= \frac{c\hbar}{2} \sum_{j,k} p^j (-2i\epsilon_{ijk}) \gamma^0 \gamma^k = -i\hbar c \gamma^0 (\mathbf{p} \times \boldsymbol{\gamma})_i; \\
[H, \mathbf{S}] &= i\hbar c \gamma^0 (\boldsymbol{\gamma} \times \mathbf{p}).
\end{aligned}
$$

Thus

$$
[H, \mathbf{J}] = [H, \mathbf{L}] + [H, \mathbf{S}] = -i\hbar c \gamma^0 (\boldsymbol{\gamma} \times \mathbf{p}) + i\hbar c \gamma^0 (\boldsymbol{\gamma} \times \mathbf{p}) = 0,
$$

and hence **J** is conserved (but not **L** or **S**).

(d)

$$
\mathbf{S}^2 = \frac{\hbar^2}{4} \boldsymbol{\Sigma}^2 = \frac{\hbar^2}{4} \begin{pmatrix} \boldsymbol{\sigma}^2 & 0 \\ 0 & \boldsymbol{\sigma}^2 \end{pmatrix}.
$$

But $\boldsymbol{\sigma}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1 + 1 + 1 = 3$, so

$$
\mathbf{S}^2 = \frac{3\hbar^2}{4}
$$

(times the unit matrix, of course). But every spinor is an eigenstate of the unit matrix (with eigenvalue 1), so every spinor is an eigenstate of $\mathbf{S}^2$, with eigenvalue $\frac{3}{4}\hbar^2$. Evidently $s = 1/2$, so the Dirac equation describes particles of spin 1/2 (no surprise for us, but it may have been for Dirac).

Problem 7.9

$$
i\gamma^2 = i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
i\gamma^2 u^{(1)*} = N \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{(E+mc^2)} \\ \frac{c(p_x - ip_y)}{(E+mc^2)} \end{pmatrix} = N \begin{pmatrix} \frac{c(p_x - ip_y)}{(E+mc^2)} \\ -\frac{cp_z}{(E+mc^2)} \\ 0 \\ 1 \end{pmatrix} = v^{(1)}.
$$

$$
i\gamma^2 u^{(2)*} = N \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x + ip_y)}{(E+mc^2)} \\ -\frac{cp_z}{(E+mc^2)} \end{pmatrix} = N \begin{pmatrix} -\frac{cp_z}{(E+mc^2)} \\ -\frac{c(p_x + ip_y)}{(E+mc^2)} \\ -1 \\ 0 \end{pmatrix} = v^{(2)}.
$$

Problem 7.10

Formally, change the sign of m (in Eq. 7.20), which changes the sign of m in Eqs. 7.23, 7.24, 7.27, 7.28 (ψ_A now represents *antiparticles*, and ψ_B particles), 7.30, 7.33, and 7.35. In Eq. 7.42 we now use the minus sign for (1) and (2), and the plus sign for (3) and (4), but the results are the same from here on. The point is that only the superficial notation is affected, not the physical content of the theory.

Problem 7.11

Following the hint, the Dirac equation in the primed frame becomes

$$i\hbar\gamma^\mu \left(\frac{\partial x^\nu}{\partial x^{\mu\prime}}\right) \partial_\nu (S\psi) - mc(S\psi) = 0.$$

Multiply on the left by S^{-1}:

$$i\hbar \left(S^{-1}\gamma^\mu S\right) \frac{\partial x^\nu}{\partial x^{\mu\prime}} \partial_\nu \psi - mc\psi = 0.$$

The Dirac equation in the original frame is

$$i\hbar\gamma^\nu \partial_\nu \psi - mc\psi = 0,$$

so solutions transform into solutions provided that

$$\left(S^{-1}\gamma^\mu S\right) \frac{\partial x^\nu}{\partial x^{\mu\prime}} = \gamma^\nu.$$

Now, the inverse Lorentz transformations (Eq. 3.3) yield (Problem 7.1)

$$\frac{\partial x^0}{\partial x^{0\prime}} = \gamma, \quad \frac{\partial x^0}{\partial x^{1\prime}} = \gamma\beta, \quad \frac{\partial x^1}{\partial x^{0\prime}} = \gamma\beta, \quad \frac{\partial x^1}{\partial x^{1\prime}} = \gamma, \quad \frac{\partial x^2}{\partial x^{2\prime}} = 1, \quad \frac{\partial x^3}{\partial x^{3\prime}} = 1,$$

and all the rest are zero. Evidently S must satisfy the following four conditions:

$$\begin{aligned}
(1)\ \gamma^0 &= \left(S^{-1}\gamma^0 S\right)\gamma + \left(S^{-1}\gamma^1 S\right)\gamma\beta \Rightarrow S\gamma^0 = (\gamma^0 S)\gamma + (\gamma^1 S)\gamma\beta, \\
(2)\ \gamma^1 &= \left(S^{-1}\gamma^0 S\right)\gamma\beta + \left(S^{-1}\gamma^1 S\right)\gamma \Rightarrow S\gamma^1 = (\gamma^0 S)\gamma\beta + (\gamma^1 S)\gamma, \\
(3)\ \gamma^2 &= \left(S^{-1}\gamma^2 S\right) \Rightarrow S\gamma^2 = \gamma^2 S, \\
(4)\ \gamma^3 &= \left(S^{-1}\gamma^3 S\right) \Rightarrow S\gamma^3 = \gamma^3 S.
\end{aligned}$$

It remains to check that $S = a_+ + a_-\gamma^0\gamma^1$ satisfies these four equations. First note that γ^0 and γ^1 anticommute with γ^2 and γ^3, and hence the *product* $\gamma^0\gamma^1$ *commutes* with them. So (3) and (4) are satisfied. As for (1) and (2):

$$\begin{aligned} S\gamma^0 &= a_+\gamma^0 + a_-\gamma^0\gamma^1\gamma^0 = a_+\gamma^0 - a_-\gamma^0\gamma^0\gamma^1 = a_+\gamma^0 - a_-\gamma^1, \\ S\gamma^1 &= a_+\gamma^1 + a_-\gamma^0\gamma^1\gamma^1 = a_+\gamma^1 - a_-\gamma^0, \\ \gamma^0 S &= a_+\gamma^0 + a_-\gamma^0\gamma^0\gamma^1 = a_+\gamma^0 + a_-\gamma^1, \\ \gamma^1 S &= a_+\gamma^1 + a_-\gamma^1\gamma^0\gamma^1 = a_+\gamma^1 - a_-\gamma^1\gamma^1\gamma^0 = a_+\gamma^1 + a_-\gamma^0. \end{aligned}$$

So equation (1) requires

$$\begin{aligned} a_+\gamma^0 - a_-\gamma^1 &= (a_+\gamma^0 + a_-\gamma^1)\gamma + (a_+\gamma^1 + a_-\gamma^0)\gamma\beta \\ &= \gamma(a_+ + \beta a_-)\gamma^0 + \gamma(a_- + \beta a_+)\gamma^1. \end{aligned}$$

Now

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \beta^2 = \frac{\gamma^2 - 1}{\gamma^2} \Rightarrow \beta = \frac{\sqrt{(\gamma-1)(\gamma+1)}}{\gamma},$$

so

$$\begin{aligned} \gamma(a_+ + \beta a_-) &= \gamma\left[\sqrt{\tfrac{1}{2}(\gamma+1)} - \frac{\sqrt{(\gamma-1)(\gamma+1)}}{\gamma}\sqrt{\tfrac{1}{2}(\gamma-1)}\right] \\ &= \sqrt{\tfrac{1}{2}(\gamma+1)}(\gamma - \gamma + 1) = a_+; \\ \gamma(a_- + \beta a_+) &= \gamma\left[-\sqrt{\tfrac{1}{2}(\gamma-1)} + \frac{\sqrt{(\gamma-1)(\gamma+1)}}{\gamma}\sqrt{\tfrac{1}{2}(\gamma+1)}\right] \\ &= \sqrt{\tfrac{1}{2}(\gamma-1)}(-\gamma + \gamma + 1) = -a_-, \end{aligned}$$

and hence (1) is satisfied. Likewise, equation (2) says

$$\begin{aligned} a_+\gamma^1 - a_-\gamma^0 &= (a_+\gamma^0 + a_-\gamma^1)\gamma\beta + (a_+\gamma^1 + a_-\gamma^0)\gamma \\ &= \gamma(a_- + \beta a_+)\gamma^0 + \gamma(a_+ + \beta a_-)\gamma^1, \end{aligned}$$

and this is confirmed by the previous two results.

Problem 7.12

If the parity transformation is to carry solutions (to the Dirac equation) into solutions, we again require

$$\left(S^{-1}\gamma^\mu S\right)\frac{\partial x^\nu}{\partial x^{\mu\prime}} = \gamma^\nu.$$

But this time $x^{\mu'} = (x^0, -x^1, -x^2, -x^2)$, so

$$\frac{\partial x^0}{\partial x^{0'}} = 1, \quad \frac{\partial x^1}{\partial x^{1'}} = -1, \quad \frac{\partial x^2}{\partial x^{2'}} = -1, \quad \frac{\partial x^3}{\partial x^{3'}} = -1,$$

and all the rest are zero. So

$$\begin{aligned}
(1)\ \gamma^0 &= \left(S^{-1}\gamma^0 S\right) \Rightarrow S\gamma^0 = \gamma^0 S,\\
(2)\ \gamma^1 &= -\left(S^{-1}\gamma^1 S\right) \Rightarrow S\gamma^1 = -\gamma^1 S,\\
(3)\ \gamma^2 &= -\left(S^{-1}\gamma^2 S\right) \Rightarrow S\gamma^2 = -\gamma^2 S,\\
(4)\ \gamma^3 &= -\left(S^{-1}\gamma^3 S\right) \Rightarrow S\gamma^3 = -\gamma^3 S.
\end{aligned}$$

Evidently S is a 4×4 matrix that commutes with γ^0, and anticommutes with the other three. The obvious solution in $S = \gamma^0$ (though you could multiply this by any number of modulus 1).

Problem 7.13

(a) From Eq. 7.53:

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}.$$

It's real, and symmetric, so $S^\dagger = S$.

$$S^\dagger S = S^2 = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 + a_-^2\sigma_1^2 & 2a_+a_-\sigma_1 \\ 2a_+a_-\sigma_1 & a_+^2 + a_-^2\sigma_1^2 \end{pmatrix}.$$

But $\sigma_1^2 = 1$, and $a_+^2 + a_-^2 = \frac{1}{2}(\gamma+1) + \frac{1}{2}(\gamma-1) = \gamma$, while

$$\begin{aligned}
2a_+a_- &= -2\cdot\frac{1}{2}\sqrt{(\gamma+1)}\sqrt{(\gamma-1)} = -\sqrt{\gamma^2-1} = -\gamma\sqrt{1-\frac{1}{\gamma^2}}\\
&= -\gamma\sqrt{1-(1-\beta^2)} = -\gamma\beta. \qquad (\beta \equiv \frac{v}{c})
\end{aligned}$$

So

$$S^\dagger S = \gamma\begin{pmatrix} 1 & -\beta\sigma_1 \\ -\beta\sigma_1 & 1 \end{pmatrix}. \quad \checkmark$$

(b)

$$S^\dagger\gamma^0 S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}$$
$$= \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ -a_-\sigma_1 & -a_+ \end{pmatrix} = \begin{pmatrix} (a_+^2 - a_-^2) & 0 \\ 0 & (a_-^2 - a_+^2) \end{pmatrix}$$

But $(a_+^2 - a_-^2) = \frac{1}{2}(\gamma+1) - \frac{1}{2}(\gamma-1) = 1$, so $S^\dagger\gamma^0 S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0$. ✓

Problem 7.14

$$(\bar\psi\gamma^5\psi)' = (\psi'^\dagger\gamma^0\gamma^5\psi') = (S\psi)^\dagger\gamma^0\gamma^5(S\psi) = \psi^\dagger S^\dagger\gamma^0\gamma^5 S\psi. \quad \text{But}$$

$$\gamma^5 S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_-\sigma_1 & a_+ \\ a_+ & a_-\sigma_1 \end{pmatrix} = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S\gamma^5.$$

So $S^\dagger\gamma^0\gamma^5 S = S^\dagger\gamma^0 S\gamma^5 = \gamma^0\gamma^5$ (Problem 7.13b), and hence

$$(\bar\psi\gamma^5\psi)' = \psi^\dagger\gamma^0\gamma^5\psi = \bar\psi\gamma^5\psi. \quad ✓$$

Problem 7.15

$$(\gamma^\mu p_\mu - mc)u = 0 \Longrightarrow u^\dagger(\gamma^{\mu\dagger}p_\mu - mc) = 0 \Longrightarrow u^\dagger(\gamma^{\mu\dagger}\gamma^0 p_\mu - \gamma^0 mc) = 0.$$

But $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$ (see below), so $u^\dagger\gamma^0(\gamma^\mu p_\mu - mc) = 0$, or $\bar u(\gamma^\mu p_\mu - mc) = 0$.

Similarly, $(\gamma^\mu p_\mu + mc)v = 0 \Rightarrow \bar v(\gamma^\mu p_\mu + mc) = 0$ (same as above, with sign of m reversed).

Proof that $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$:

$\gamma^{0\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0$, so it holds for $\mu = 0$.

$$(\gamma^i)^\dagger = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -(\sigma^i)^\dagger \\ (\sigma^i)^\dagger & 0 \end{pmatrix}.$$

But $(\sigma^i)^\dagger = \sigma^i$:

$$(\sigma^1)^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1;$$

$$(\sigma^2)^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^2;$$

$$(\sigma^3)^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3.$$

So $(\gamma^i)^\dagger = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} = -\gamma^i$. But γ^i anticommutes with γ^0, so $\gamma^i\gamma^0 = -\gamma^0\gamma^i$. Therefore $(\gamma^i)^\dagger\gamma^0 = -\gamma^i\gamma^0 = \gamma^0\gamma^i$. So it holds for $\mu = i = 1, 2, 3$ also. ✓

Problem 7.16

In the notation of Eq. 7.42 and 7.46,

$$\bar{u}u = u^\dagger\gamma^0 u = N^2 \left(u_A^\dagger \;\; u_B^\dagger\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = N^2 \left(u_A^\dagger u_A - u_B^\dagger u_B\right).$$

In particular, for $u^{(1)}$:

$$\begin{aligned}
\bar{u}u &= N^2 \left[(1 \;\; 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{c^2}{(E+mc^2)^2} \left(p_z \;\; (p_x - ip_y)\right) \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \right] \\
&= N^2 \left[1 - \frac{c^2}{(E+mc^2)^2} (p_z^2 + p_x^2 + p_y^2) \right] = \frac{N^2}{(E+mc^2)^2} \left[(E+mc^2)^2 - c^2\mathbf{p}^2 \right] \\
&= \frac{(E+mc^2)}{c} \frac{1}{(E+mc^2)^2} \left[E^2 + 2Emc^2 + m^2c^4 - c^2\mathbf{p}^2 \right] \\
&= \frac{1}{c(E+mc^2)} \left(2Emc^2 + 2m^2c^4 \right) = \frac{1}{c(E+mc^2)} 2mc^2(E+mc^2) = 2mc. \checkmark
\end{aligned}$$

It's essentially the same for $u^{(2)}$; for $v^{(1)}$ and $v^{(2)}$ we use Eq. 7.47:

$$\begin{aligned}
\bar{v}v &= N^2 \left[\frac{c^2}{(E+mc^2)^2} \left((p_x + ip_y) \;\; -p_z\right) \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} - (0 \;\; 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\
&= N^2 \left[\frac{c^2}{(E+mc^2)^2} (p_x^2 + p_y^2 + p_z^2) - 1 \right] = \frac{N^2}{(E+mc^2)^2} \left[c^2\mathbf{p}^2 - (E+mc^2)^2 \right] \\
&= \frac{(E+mc^2)}{c} \frac{1}{(E+mc^2)^2} \left[c^2\mathbf{p}^2 - E^2 - 2Emc^2 - m^2c^4 \right] \\
&= \frac{-1}{c(E+mc^2)} \left(2Emc^2 + 2m^2c^4 \right) = \frac{-1}{c(E+mc^2)} 2mc^2(E+mc^2) = -2mc. \checkmark
\end{aligned}$$

Problem 7.17

Let $a^\mu \equiv \bar{\psi}\gamma^\mu\psi$. Then

$$a^{0\prime} = (\bar{\psi}\gamma^0\psi)' = \psi'^\dagger\gamma^0\gamma^0\psi' = (S\psi)^\dagger\gamma^0\gamma^0(S\psi) = \psi^\dagger S^\dagger\gamma^0\gamma^0 S\psi.$$

But $\gamma^0\gamma^0 = 1$, so $(\bar{\psi}\gamma^0\psi)' = \psi^\dagger(S^\dagger S)\psi$.

$$\begin{aligned} S^\dagger S &= \gamma\begin{pmatrix} 1 & -\beta\sigma_1 \\ -\beta\sigma_1 & 1 \end{pmatrix} \quad \text{(Problem 7.13a)} \\ &= \gamma - \beta\gamma\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \gamma - \beta\gamma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\ &= \gamma(\gamma^0\gamma^0) - \beta\gamma(\gamma^0\gamma^1) = \gamma^0(\gamma\gamma^0 - \beta\gamma\gamma^1). \end{aligned}$$

$$a^{0\prime} = \psi^\dagger\gamma^0(\gamma\gamma^0 - \beta\gamma\gamma^1)\psi = \bar{\psi}(\gamma\gamma^0 - \beta\gamma\gamma^1)\psi = \gamma a^0 - \beta\gamma a^1. \checkmark$$

Similarly

$$a^{1\prime} = (\bar{\psi}\gamma^1\psi)' = \psi^\dagger S^\dagger\gamma^0\gamma^1 S\psi.$$

$$\begin{aligned} S^\dagger\gamma^0\gamma^1 S &= (a_+ + a_-\gamma^0\gamma^1)\gamma^0\gamma^1(a_+ + a_-\gamma^0\gamma^1) \\ &= a_+^2\gamma^0\gamma^1 + 2a_+a_-(\gamma^0\gamma^1)^2 + a_-^2(\gamma^0\gamma^1)^3. \end{aligned}$$

$$(\gamma^0\gamma^1)^2 = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\right]^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma_1 & 0 \end{pmatrix}^2 = \begin{pmatrix} (\sigma^1)^2 & 0 \\ 0 & (\sigma^1)^2 \end{pmatrix} = 1.$$

$$\begin{aligned} S^\dagger\gamma^0\gamma^1 S &= (a_+^2 + a_-^2)\gamma^0\gamma^1 + 2a_+a_- = \gamma\gamma^0\gamma^1 - \gamma\beta\gamma^0\gamma^0 \\ &= \gamma^0(\gamma\gamma^1 - \beta\gamma\gamma^0) \quad \text{(using results from Problem 7.13a).} \end{aligned}$$

$$a^{1\prime} = \psi^\dagger\gamma^0(\gamma\gamma^1 - \beta\gamma\gamma^0)\psi = \bar{\psi}(\gamma\gamma^1 - \beta\gamma\gamma^0)\psi = \gamma a^1 - \beta\gamma a^0. \checkmark$$

Finally,

$$a^{2\prime} = (\bar{\psi}\gamma^2\psi)' = \psi^\dagger S^\dagger\gamma^0\gamma^2 S\psi.$$

$$\gamma^2 S = \gamma^2(a_+ + a_-\gamma^0\gamma^1) = (a_+ + a_-\gamma^0\gamma^1)\gamma^2 = S\gamma^2$$

(since γ^2 anticommutes with γ^0 and γ^1).

$$a^{2\prime} = \psi^\dagger S^\dagger\gamma^0 S\gamma^2\psi = \psi^\dagger\gamma^0\gamma^2\psi = \bar{\psi}\gamma^2\psi = a^2. \quad \checkmark$$

(I used $S^\dagger\gamma^0 S = \gamma^0$.) Same for a^3.
Under parity, $\psi' = \gamma^0\psi$ (Eq. 7.61), so

$$\begin{aligned} a^{0\prime} &= (\bar\psi\gamma^0\psi)' = \psi^\dagger\gamma^0\gamma^0\gamma^0\gamma^0\psi = \psi^\dagger(\gamma^0)^2\psi && \text{(because } (\gamma^0)^2 = 1\text{)} \\ &= \bar\psi\gamma^0\psi = a^0 && \text{(so } a^0 \text{ is invariant under } P\text{)} \\ a^{i\prime} &= (\bar\psi\gamma^i\psi)^\dagger = \psi^\dagger\gamma^0\gamma^0\gamma^i\gamma^0\psi = \psi^\dagger\gamma^i\gamma^0\psi \\ &= \psi^\dagger(-\gamma^0\gamma^i)\psi = -\bar\psi\gamma^i\psi = -a^i && \text{(so "spatial" parts change sign).} \end{aligned}$$

Problem 7.18

Equation 7.61 $\Longrightarrow$ $P\psi = \gamma^0\psi$, so $P\psi^{(1)} = \psi^{(1)}$, $P\psi^{(2)} = \psi^{(2)}$, $P\psi^{(3)} = -\psi^{(3)}$, and $P\psi^{(4)} = -\psi^{(4)}$. Evidently $\psi^{(1)}$ and $\psi^{(2)}$ (the electron states) are eigenstates of P with eigenvalue $+1$ (*positive* intrinsic parity), while $\psi^{(3)}$ and $\psi^{(4)}$ (the positron states) have eigenvalue -1. If $P = -\gamma^0$, the parities are reversed, but it is still the case that particle and antiparticle have *opposite* parity.

Problem 7.19

(a) From Eqs. 7.15 and 7.69:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu},$$
$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \Rightarrow \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = -2i\sigma^{\mu\nu}.$$

Adding,

$$2\gamma^\mu\gamma^\nu = 2\left(g^{\mu\nu} - i\sigma^{\mu\nu}\right), \quad \text{or} \quad \boxed{\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}}.$$

(b)

$$\begin{aligned} \sigma^{12} &= \tfrac{i}{2}\left(\gamma^1\gamma^2 - \gamma^2\gamma^1\right) = i\gamma^1\gamma^2 = i\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \\ &= i\begin{pmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & -\sigma^1\sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \Sigma^3. \end{aligned}$$

I used the anticommutation relation for the gamma matrices, and $\sigma^1\sigma^2 = i\sigma^3$ (Problem 4.19); note that because σ_i is not part of a 4-vector, we do not distinguish upper and lower indices.

Similarly,

$$\begin{aligned}\sigma^{13} &= \tfrac{i}{2}\left(\gamma^1\gamma^3 - \gamma^3\gamma^1\right) = i\gamma^1\gamma^3 = i\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0\end{pmatrix} \\ &= i\begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3\end{pmatrix} = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & -\sigma^2\end{pmatrix} = -\Sigma^2.\end{aligned}$$

$$\begin{aligned}\sigma^{23} &= \tfrac{i}{2}\left(\gamma^2\gamma^3 - \gamma^3\gamma^2\right) = i\gamma^2\gamma^3 = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0\end{pmatrix} \\ &= i\begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3\end{pmatrix} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1\end{pmatrix} = \Sigma^1.\end{aligned}$$

Problem 7.20

(a) Start with Eq. 7.73: $\partial_\mu F^{\mu\nu} = (4\pi/c)J^\nu$. For the case $\nu = 0$:

$$\begin{aligned}\partial_\mu F^{\mu 0} &= \frac{\partial}{\partial x^0}F^{00} + \frac{\partial}{\partial x^1}F^{10} + \frac{\partial}{\partial x^2}F^{20} + \frac{\partial}{\partial x^3}F^{30} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \nabla\cdot\mathbf{E} \\ &= \frac{4\pi}{c}J^0 = \frac{4\pi}{c}c\rho = 4\pi\rho. \quad \text{So} \quad \nabla\cdot\mathbf{E} = 4\pi\rho. \checkmark\end{aligned}$$

For the case $\nu = 1$:

$$\begin{aligned}\partial_\mu F^{\mu 1} &= \frac{\partial}{\partial x^0}F^{01} + \frac{\partial}{\partial x^1}F^{11} + \frac{\partial}{\partial x^2}F^{21} + \frac{\partial}{\partial x^3}F^{31} = \frac{\partial(-E_x)}{\partial(ct)} + \frac{\partial B_z}{\partial y} + \frac{\partial(-B_y)}{\partial z} \\ &= -\frac{1}{c}\frac{\partial E_x}{\partial t} + \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = \left[-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + (\nabla\times\mathbf{B})\right]_x \\ &= \frac{4\pi}{c}J^1 = \frac{4\pi}{c}J_x.\end{aligned}$$

This is the x component of

$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \quad \checkmark$$

(the y component comes from $\nu = 2$, and the z component from $\nu = 3$).

(b) Take the divergence of Eq. 7.73:

$$\partial_\nu\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\partial_\nu J^\nu.$$

But $\partial_\nu\partial_\mu$ is symmetric in $\mu \leftrightarrow \nu$ (by the equality of cross-derivatives), whereas $F^{\mu\nu}$ is antisymmetric, so the left side is zero (Problem 3.10e), and hence $\partial_\mu J^\mu = 0$. ✓

Problem 7.21

The continuity equation (Eq. 7.74) says

$$0 = \partial_\mu J^\mu = \partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 = \frac{\partial(c\rho)}{\partial(ct)} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$
$$\Longrightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.$$

Integrate over some volume V, with surface S:

$$\int_V \frac{\partial \rho}{\partial t} d\tau = \frac{d}{dt}\int_V \rho\, d\tau = \frac{dQ}{dt} = -\int_V \nabla \cdot \mathbf{J}\, d\tau = -\int_S \mathbf{J} \cdot d\mathbf{A}$$

(where Q is the total charge in V, and I used the divergence theorem in the last step). This says that the rate of change of the charge in V is minus the flux of charge out through the surface—no charge simply disappears or is created from nothing. In particular, if we pick a volume such that $\mathbf{J}$ is *zero* at the surface, then Q is constant.

Problem 7.22

$$A_0' = A_0 + \partial_0 \lambda = 0 \quad \Rightarrow \quad \partial_0 \lambda = -A_0 \quad \Rightarrow \quad \frac{\partial \lambda}{\partial t} = -cA_0.$$

$$\boxed{\lambda(\mathbf{r}, t) = -c\int_{-\infty}^{t} A_0(\mathbf{r}, t')\, dt'}.$$

(You don't have to use $-\infty$ as the lower limit—any constant would do.)

$$\begin{aligned}\Box\lambda &= \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2} - \nabla^2 \lambda = -\frac{1}{c}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{t} A_0(\mathbf{r}, t')\, dt' + c\nabla^2 \int_{-\infty}^{t} A_0(\mathbf{r}, t')\, dt' \\ &= -\frac{1}{c}\frac{\partial}{\partial t}\left[A_0(\mathbf{r}, t)\right] + c\int_{-\infty}^{t} \nabla^2 A_0(\mathbf{r}, t')\, dt'.\end{aligned}$$

But $\Box A^\mu = 0 \Rightarrow \nabla^2 A_0(\mathbf{r}, t') = (1/c^2)(\partial^2/\partial t'^2) A_0(\mathbf{r}, t')$, so

$$\begin{aligned}\Box\lambda &= -\frac{1}{c}\frac{\partial A_0}{\partial t} + \frac{1}{c}\left[\frac{\partial}{\partial t'} A_0(\mathbf{r}, t')\right]\Bigg|_{-\infty}^{t} \\ &= -\frac{1}{c}\frac{\partial A_0}{\partial t} + \frac{1}{c}\frac{\partial A_0}{\partial t} - \frac{1}{c}\frac{\partial A_0(\mathbf{r}, t')}{\partial t'}\Bigg|_{-\infty} = -\frac{1}{c}\frac{\partial A_0(\mathbf{r}, t')}{\partial t'}\Bigg|_{-\infty} = 0. \checkmark\end{aligned}$$

Problem 7.23

(a)

$$\Box\lambda = \partial^\mu \partial_\mu \lambda = i\hbar\kappa a\, \partial^\mu \partial_\mu\, e^{-ip\cdot x/\hbar} = i\hbar\kappa a \left(-\frac{i}{\hbar}\right)^2 p^\mu p_\mu\, e^{-ip\cdot x/\hbar} = 0 \quad \checkmark$$

(Eq. 7.90).

(b)

$$\begin{aligned} A^{\mu\prime} &= A^\mu + \partial^\mu \lambda = a e^{-ip\cdot x/\hbar} \epsilon^\mu + \partial^\mu \left(i\hbar\kappa a e^{-ip\cdot x/\hbar}\right) \\ &= a e^{-ip\cdot x/\hbar} \epsilon^\mu + i\hbar\kappa a \left(-\frac{i}{\hbar} p^\mu\right) e^{-ip\cdot x/\hbar} = a e^{-ip\cdot x/\hbar} \left(\epsilon^\mu + \kappa p^\mu\right). \end{aligned}$$

Thus $\epsilon^{\mu\prime} = \epsilon^\mu + \kappa p^\mu$.

Problem 7.24

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = |N|^2 \left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{cp_z}{E+mc^2} & -\frac{c(p_x-ip_y)}{E+mc^2} \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ -\frac{cp_z}{E+mc^2} \end{pmatrix} \begin{pmatrix} 0 & 1 & -\frac{c(p_x+ip_y)}{E+mc^2} & \frac{cp_z}{E+mc^2} \end{pmatrix} \right\}$$
$$= \left(\frac{E+mc^2}{c}\right) \left\{ \begin{pmatrix} 1 & 0 & \frac{-cp_z}{E+mc^2} & \frac{-c(p_x-ip_y)}{E+mc^2} \\ 0 & 0 & 0 & 0 \\ \frac{cp_z}{E+mc^2} & 0 & \frac{-c^2p_z^2}{(E+mc^2)^2} & \frac{-c^2p_z(p_x-ip_y)}{(E+mc^2)^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} & 0 & \frac{-c^2p_z(p_x+ip_y)}{(E+mc^2)^2} & \frac{-c^2(p_x^2+p_y^2)}{(E+mc^2)^2} \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{-c(p_x+ip_y)}{E+mc^2} & \frac{cp_z}{E+mc^2} \\ 0 & \frac{c(p_x-ip_y)}{E+mc^2} & \frac{-c^2(p_x^2+p_y^2)}{(E+mc^2)^2} & \frac{c^2p_z(p_x-ip_y)}{(E+mc^2)^2} \\ 0 & \frac{-cp_z}{E+mc^2} & \frac{c^2p_z(p_x+ip_y)}{(E+mc^2)^2} & \frac{-c^2p_z^2}{(E+mc^2)^2} \end{pmatrix} \right\}$$
$$= \begin{pmatrix} \frac{E+mc^2}{c} & 0 & -p_z & -(p_x-ip_y) \\ 0 & \frac{E+mc^2}{c} & -(p_x+ip_y) & p_z \\ p_z & (p_x-ip_y) & -\frac{c\mathbf{p}^2}{E+mc^2} & 0 \\ (p_x+ip_y) & -p_z & 0 & -\frac{c\mathbf{p}^2}{E+mc^2} \end{pmatrix}.$$

But $\mathbf{p}\cdot\boldsymbol{\sigma} = \begin{pmatrix} p_z & p_x-ip_y \\ p_x+ip_y & -p_z \end{pmatrix}$ and $\dfrac{c^2\mathbf{p}^2}{E+mc^2} = \dfrac{E^2-m^2c^4}{E+mc^2} = E-mc^2$, so

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = \begin{pmatrix} \begin{pmatrix} \frac{E}{c}+mc & 0 \\ 0 & \frac{E}{c}+mc \end{pmatrix} & -(\mathbf{p}\cdot\sigma) \\ (\mathbf{p}\cdot\sigma) & \begin{pmatrix} -\frac{E}{c}+mc & 0 \\ 0 & -\frac{E}{c}+mc \end{pmatrix} \end{pmatrix}$$
$$= \frac{E}{c}\gamma^0 - \mathbf{p}\cdot\begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} + mc = \gamma^\mu p_\mu + mc. \quad \checkmark$$

$$\sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = |N|^2 \left\{ \begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ -\frac{cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{c(p_x+ip_y)}{E+mc^2} & -\frac{cp_z}{E+mc^2} & 0 & -1 \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2} & -1 & 0 \end{pmatrix} \right\}$$

$$= \left(\frac{E+mc^2}{c}\right) \left\{ \begin{pmatrix} \frac{c^2(p_x^2+p_y^2)}{(E+mc^2)^2} & \frac{-c^2p_z(p_x-ip_y)}{(E+mc^2)^2} & 0 & \frac{-c(p_x-ip_y)}{E+mc^2} \\ \frac{-c^2p_z(p_x+ip_y)}{(E+mc^2)^2} & \frac{c^2p_z^2}{(E+mc^2)^2} & 0 & \frac{cp_z}{E+mc^2} \\ 0 & 0 & 0 & 0 \\ \frac{c(p_x+ip_y)}{E+mc^2} & \frac{-cp_z}{E+mc^2} & 0 & -1 \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \frac{c^2p_z^2}{(E+mc^2)^2} & \frac{c^2p_z(p_x-ip_y)}{(E+mc^2)^2} & -\frac{cp_z}{E+mc^2} & 0 \\ \frac{c^2p_z(p_x+ip_y)}{(E+mc^2)^2} & \frac{c^2(p_x^2+p_y^2)}{(E+mc^2)^2} & \frac{-c(p_x+ip_y)}{E+mc^2} & 0 \\ \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \frac{c\mathbf{p}^2}{E+mc^2} & 0 & -p_z & -(p_x-ip_y) \\ 0 & \frac{c\mathbf{p}^2}{E+mc^2} & -(p_x+ip_y) & p_z \\ p_z & (p_x-ip_y) & -\frac{E+mc^2}{c} & 0 \\ p_x+ip_y & -p_z & 0 & -\frac{E+mc^2}{c} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \frac{E-mc^2}{c} & 0 \\ 0 & \frac{E-mc^2}{c} \end{pmatrix} & -(\mathbf{p}\cdot\sigma) \\ (\mathbf{p}\cdot\sigma) & \begin{pmatrix} -\frac{E+mc^2}{c} & 0 \\ 0 & -\frac{E+mc^2}{c} \end{pmatrix} \end{pmatrix}$$

$$= \frac{E}{c}\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma} - mc = \gamma^\mu p_\mu - mc. \quad \checkmark$$

Problem 7.25

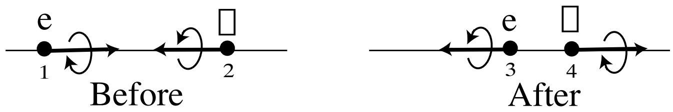
$$\boldsymbol{\epsilon}^{(1)} = (1,0,0), \quad \boldsymbol{\epsilon}^{(2)} = (0,1,0), \quad \mathbf{p} = (0,0,p) \Rightarrow \hat{\mathbf{p}} = (0,0,1).$$

$$\sum_{s=1,2} \epsilon_i^{(s)}\epsilon_j^{(s)*} = \epsilon_i^{(1)}\epsilon_j^{(1)*} + \epsilon_i^{(2)}\epsilon_j^{(2)*}$$

is zero unless $i = j$; it is 1 for $i = j = 1$ and $i = j = 2$, but 0 for $i = j = 3$, so δ_{ij} doesn't quite do it, but if we subtract off $\hat{p}_i\hat{p}_j$, which is 1 for $i = j = 3$ and 0 otherwise, that will fix the problem:

$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j. \quad \checkmark$$

Problem 7.26



From Problem 7.6,

$$u(1) = \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix}, \quad u(2) = \begin{pmatrix} 0 \\ b_+ \\ 0 \\ b_- \end{pmatrix}, quad u(3) = \begin{pmatrix} 0 \\ a_+ \\ 0 \\ a_- \end{pmatrix}, \quad u(4) = \begin{pmatrix} b_+ \\ 0 \\ b_- \\ 0 \end{pmatrix},$$

where $a_\pm \equiv \sqrt{(E_e \pm mc^2)/c}$; $\quad b_\pm \equiv \sqrt{(E_\mu \pm Mc^2)/c}$. [*Note:* The bottom entry in $u(2)$ is *not* $-b_-$ (nor is it $-a_-$ in $u(3)$) – unlike Problem 7.6. The point is that (2) and (3) are going in the $-z$ direction, so $p_z = -|\mathbf{p}|$ for them.] From Eq. 7.106,

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2} \left\{ [\bar{u}(3)\gamma^0 u(1)][\bar{u}(4)\gamma^0 u(2)] - [\bar{u}(3)\gamma^i u(1)][\bar{u}(4)\gamma^i u(2)] \right\}$$

(where i is summed from 1 to 3).

$$\bar{u}(3)\gamma^0 u(1) = \begin{pmatrix} 0 & a_+ & 0 & a_- \end{pmatrix} \gamma^0 \gamma^0 \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & a_+ & 0 & a_- \end{pmatrix} \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix} = 0.$$

$$\bar{u}(3)\gamma^i u(1) = \begin{pmatrix} 0 & a_+ & 0 & a_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_+ & 0 & -a_- \end{pmatrix} \begin{pmatrix} \sigma^i \begin{pmatrix} a_- \\ 0 \end{pmatrix} \\ -\sigma^i \begin{pmatrix} a_+ \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & a_+ \end{pmatrix} \sigma^i \begin{pmatrix} a_- \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & a_- \end{pmatrix} \sigma^i \begin{pmatrix} a_+ \\ 0 \end{pmatrix}$$

$$= 2a_+ a_- \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11}^i & \sigma_{12}^i \\ \sigma_{21}^i & \sigma_{22}^i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = 2a_+ a_- \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11}^i \\ \sigma_{21}^i \end{pmatrix} \right] = 2a_+ a_- \sigma_{21}^i.$$

$$\bar{u}(4)\gamma^i u(2) = \begin{pmatrix} b_+ & 0 & b_- & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ b_+ \\ 0 \\ b_- \end{pmatrix}$$

$$= \begin{pmatrix} b_+ & 0 & -b_- & 0 \end{pmatrix} \begin{pmatrix} \sigma^i \begin{pmatrix} 0 \\ b_- \end{pmatrix} \\ -\sigma^i \begin{pmatrix} 0 \\ b_+ \end{pmatrix} \end{pmatrix} = \begin{pmatrix} b_+ & 0 \end{pmatrix} \sigma^i \begin{pmatrix} 0 \\ b_- \end{pmatrix} + \begin{pmatrix} b_- & 0 \end{pmatrix} \sigma^i \begin{pmatrix} 0 \\ b_+ \end{pmatrix}$$

$$= 2b_+b_- \left[(1\ 0) \begin{pmatrix} \sigma^i_{11} & \sigma^i_{12} \\ \sigma^i_{21} & \sigma^i_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = 2b_+b_- \left[(1\ 0) \begin{pmatrix} \sigma^i_{12} \\ \sigma^i_{22} \end{pmatrix} \right] = 2b_+b_-\sigma^i_{12}.$$

$$\mathcal{M} = \frac{g_e^2}{(p_1 - p_3)^2}(2a_+a_- 2b_+b_-)\boldsymbol{\sigma}_{21} \cdot \boldsymbol{\sigma}_{12} = \frac{8g_e^2}{(p_1 - p_3)^2}(a_+a_-)(b_+b_-),$$

where I used $\boldsymbol{\sigma}_{21} \cdot \boldsymbol{\sigma}_{12} = (1)(1) + (i)(-i) + (0)(0) = 2$ in the last step. Now

$$(a_+a_-) = \sqrt{\frac{E_e^2 - m^2c^4}{c^2}} = \sqrt{\frac{\mathbf{p}_e^2 c^2}{c^2}} = |\mathbf{p}_e|, \quad (b_+b_-) = |\mathbf{p}_\mu|, \text{ and } |\mathbf{p}_e| = |\mathbf{p}_\mu|.$$

$$\text{So } \mathcal{M} = \frac{8g_e^2\mathbf{p}_e^2}{(p_1 - p_3)^2}.$$

$$p_1 = \left(\frac{E_e}{c}, \mathbf{p}_e\right),\ p_3 = \left(\frac{E_e}{c}, -\mathbf{p}_e\right);\ \text{so } (p_1 - p_3) = (0, 2\mathbf{p}_e),\ (p_1 - p_3)^2 = 0 - 4\mathbf{p}_e^2.$$

$$\therefore \mathcal{M} = \frac{8g_e^2\mathbf{p}_e^2}{-4\mathbf{p}_e^2} = \boxed{-2g_e^2.}$$

Problem 7.27

There are two diagrams:

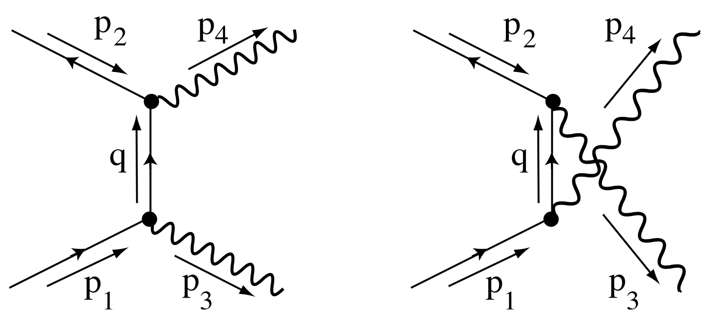


Diagram 1:

$$\int \epsilon_\mu^*(4)\left[\bar{v}(2)(ig_e\gamma^\mu)\frac{i(\not{q}+mc)}{q^2-m^2c^2}(ig_e\gamma^\nu)u(1)\right]\epsilon_\nu^*(3)$$
$$\times (2\pi)^4\delta^4(p_1-q-p_3)(2\pi)^4\delta^4(p_2+q-p_4)\,\frac{d^4q}{(2\pi)^4}$$
$$= -ig_e^2\left[\bar{v}(2)\not{\epsilon}^*(4)\frac{(\not{p}_1-\not{p}_3+mc)}{(p_1-p_3)^2-m^2c^2}\not{\epsilon}^*(3)u(1)\right]$$
$$\times \underbrace{(2\pi)^4\delta^4(p_2+p_1-p_3-p_4)}_{\text{erase}}$$

$$\mathcal{M}_1 = \boxed{\frac{g_e^2}{(p_1-p_3)^2-m^2c^2}\left[\bar{v}(2)\not{\epsilon}^*(4)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(3)u(1)\right]}.$$

Diagram 2: The same as **Diagram 1**, only with 3↔4. So

$$\mathcal{M}_2 = \boxed{\frac{g_e^2}{(p_1-p_4)^2-m^2c^2}\left[\bar{v}(2)\not{\epsilon}^*(3)(\not{p}_1-\not{p}_4+mc)\not{\epsilon}^*(4)u(1)\right]}.$$

Problem 7.28

$$[\bar{v}(a)\Gamma_2 v(b)]^* = \left[v(a)^\dagger\gamma^0\Gamma_2 v(b)\right]^\dagger = v(b)^\dagger\Gamma_2^\dagger\gamma^{0\dagger}v(a)$$
$$= v(b)^\dagger\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 v(a) = \bar{v}(b)\bar{\Gamma}_2 v(a)$$

$$\sum_{\text{spins}}[\bar{v}(a)\Gamma_1 v(b)]\,[\bar{v}(a)\Gamma_2 v(b)]^* = \sum_{s(a)}\bar{v}(a)\Gamma_1\left\{\sum_{s(b)}v(b)\bar{v}(b)\right\}\bar{\Gamma}_2 v(a).$$

Invoking the completeness relation (Eq. 7.99),

$$\sum_{s(b)}v(b)\bar{v}(b) = (\not{p}_b - m_b c)\,,$$

and defining $Q \equiv \Gamma_1(\not{p}_b - m_b c)\bar{\Gamma}_2$,

$$\sum_{\text{spins}}[\bar{v}(a)\Gamma_1 v(b)]\,[\bar{v}(a)\Gamma_2 v(b)]^* = \sum_{s(a)}\bar{v}(a)Qv(a) = \sum_{s(a)}\sum_{i,j}\bar{v}(a)_i Q_{ij} v(a)_j$$
$$= \sum_{i,j}Qij\left\{\sum_{s(a)}v(a)_j\bar{v}(a)_i\right\} = \sum_{i,j}Q_{ij}\,(\not{p}_a - m_a c)_{ji} = \sum_i\,[Q(\not{p}_a - m_a c)]_{ii}$$
$$= \text{Tr}\,[Q(\not{p}_a - m_a c)] = \boxed{\text{Tr}\,[\Gamma_1(\not{p}_b - m_b c)\bar{\Gamma}_2(\not{p}_a - m_a c)]}.$$

The others are identical, except that the completeness relation carries a plus sign for particle states. So

$$\sum_{\text{spins}} [\bar{u}(a)\Gamma_1 v(b)]\,[\bar{u}(a)\Gamma_2 v(b)]^* = \boxed{\text{Tr}\left[\Gamma_1(\not{p}_b - m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)\right]},$$

$$\sum_{\text{spins}} [\bar{v}(a)\Gamma_1 u(b)]\,[\bar{v}(a)\Gamma_2 u(b)]^* = \boxed{\text{Tr}\left[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a - m_a c)\right]}.$$

Problem 7.29

(a) If $\nu = 0$, note that $\gamma^{0\dagger} = \gamma^0$ and $\gamma^0\gamma^0 = 1$, so $\gamma^0\gamma^{0\dagger}\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$. ✓
If $\nu = i = 1, 2, 3$, then $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \Longrightarrow (\gamma^i)^\dagger = \begin{pmatrix} 0 & -\sigma^{i\dagger} \\ \sigma^{i\dagger} & 0 \end{pmatrix}$. But $\sigma^{i\dagger} = \sigma^i$, so $(\gamma^i)^\dagger = -\gamma^i$. Moreover, $\gamma^i\gamma^0 = -\gamma^0\gamma^i$, so $\gamma^0\gamma^{i\dagger}\gamma^0 = -\gamma^0\gamma^i\gamma^0 = \gamma^0\gamma^0\gamma^i = \gamma^i$. ✓

(b)

$$\bar{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0 = \gamma^0(\gamma_a\gamma_b\cdots\gamma_c)^\dagger\gamma^0 = \gamma^0(\gamma_c^\dagger\cdots\gamma_b^\dagger\gamma_a^\dagger)\gamma^0.$$

Insert $\gamma^0\gamma^0 = 1$ between every pair:

$$\begin{aligned}\bar{\Gamma} &= \gamma^0\left[\gamma_c^\dagger(\gamma^0\gamma^0)\cdots(\gamma^0\gamma^0)\gamma_b^\dagger(\gamma^0\gamma^0)\gamma_a^\dagger\right]\gamma^0 \\ &= (\gamma^0\gamma_c^\dagger\gamma^0)\cdots(\gamma^0\gamma_b^\dagger\gamma^0)(\gamma^0\gamma_a^\dagger\gamma^0) = \gamma_c\cdots\gamma_b\gamma_a. \checkmark\end{aligned}$$

Problem 7.30

According to Eq. 7.112, $\mathcal{M}_1 = A\,[\bar{u}(4)\Gamma_1 u(1)]$, where

$$A \equiv \frac{g_e^2}{(p_1 - p_3)^2 - (mc)^2} \quad \text{and} \quad \Gamma_1 \equiv \not{\epsilon}(2)(\not{p}_1 - \not{p}_3 + mc)\not{\epsilon}(3)^*.$$

First sum over the outgoing electron spins (s_4) and average over the incoming electron spins (s_1). Casimir's trick (Eq. 7.125) gives $\langle|\mathcal{M}_1|^2\rangle_e$

$$= \frac{1}{2}A^2\sum_{s_1,s_4}[\bar{u}(4)\Gamma_1 u(1)]\,[\bar{u}(4)\Gamma_1 u(1)]^* = \frac{A^2}{2}\text{Tr}\left[\Gamma_1(\not{p}_1 + mc)\bar{\Gamma}_1(\not{p}_4 + mc)\right],$$

where

$$\begin{aligned}\bar{\Gamma}_1 &\equiv \gamma^0\Gamma_1^\dagger\gamma^0 = \gamma^0\left[\not{\epsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}(3)^*\right]^\dagger\gamma^0\\ &= \gamma^0\left[\not{\epsilon}(3)^\dagger(\not{p}_1^\dagger-\not{p}_3^\dagger+mc)\not{\epsilon}(2)^{*\dagger}\right]\gamma^0\\ &= \left[\gamma^0\not{\epsilon}(3)^\dagger\gamma^0\right]\left[\gamma^0(\not{p}_1^\dagger-\not{p}_3^\dagger+mc)\gamma^0\right]\left[\gamma^0\not{\epsilon}(2)^{*\dagger}\gamma^0\right]\\ &= \not{\epsilon}(3)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}(2)^*.\end{aligned}$$

Similarly, by Eq. 7.113, $\mathcal{M}_2 = B\left[\bar{u}(4)\Gamma_2 u(1)\right]$, where

$$B \equiv \frac{g_e^2}{(p_1+p_2)^2-(mc)^2} \quad \text{and} \quad \Gamma_2 \equiv \not{\epsilon}(3)^*(\not{p}_1+\not{p}_2+mc)\not{\epsilon}(2).$$

$$\langle|\mathcal{M}_2|^2\rangle_e = \frac{B^2}{2}\mathrm{Tr}\left[\Gamma_2(\not{p}_1+mc)\bar{\Gamma}_2(\not{p}_4+mc)\right],\ \bar{\Gamma}_2 = \not{\epsilon}(2)^*(\not{p}_1+\not{p}_2+mc)\not{\epsilon}(3).$$

$$\begin{aligned}\langle\mathcal{M}_1\mathcal{M}_2^*\rangle_e &= \frac{AB}{2}\sum_{s_1,s_4}\left[\bar{u}(4)\Gamma_1 u(1)\right]\left[\bar{u}(4)\Gamma_2 u(1)\right]^*\\ &= \frac{AB}{2}\mathrm{Tr}\left[\Gamma_1(\not{p}_1+mc)\bar{\Gamma}_2(\not{p}_4+mc)\right],\\ \langle\mathcal{M}_2\mathcal{M}_1^*\rangle_e &= \frac{AB}{2}\sum_{s_1,s_4}\left[\bar{u}(4)\Gamma_2 u(1)\right]\left[\bar{u}(4)\Gamma_1 u(1)\right]^*\\ &= \frac{AB}{2}\mathrm{Tr}\left[\Gamma_2(\not{p}_1+mc)\bar{\Gamma}_1(\not{p}_4+mc)\right].\end{aligned}$$

Now sum and average over the photon spins:

$$\begin{aligned}\langle|\mathcal{M}_1|^2\rangle &= \frac{A^2}{4}\sum_{s_2,s_3}\mathrm{Tr}\Big[\epsilon(2)_\mu\gamma^\mu(\not{p}_1-\not{p}_3+mc)\epsilon(2)^*_\nu\gamma^\nu(\not{p}_1+mc)\epsilon(3)_\kappa\gamma^\kappa\\ &\qquad\times(\not{p}_1-\not{p}_3+mc)\epsilon(2)^*_\lambda\gamma^\lambda(\not{p}_4+mc)\Big].\end{aligned}$$

Writing the completeness relation for photons (Eq. 7.105) as

$$\sum_{s_1,s_2}\epsilon_\mu^{(s)}\epsilon_\nu^{(s)} = Q_{\mu\nu}, \quad \text{where} \quad Q_{\mu\nu} \equiv \begin{cases}0, & \text{if } \mu \text{ or } \nu \text{ is } 0\\ \delta_{ij}-\hat{p}_i\hat{p}_j, & \text{otherwise}\end{cases}$$

$$\begin{aligned}\langle|\mathcal{M}_1|^2\rangle &= \frac{A^2}{4}Q_{\mu\lambda}Q_{\nu\kappa}\mathrm{Tr}\Big[\gamma^\mu(\not{p}_1-\not{p}_3+mc)\gamma^\nu(\not{p}_1+mc)\gamma^\kappa\\ &\qquad\times(\not{p}_1-\not{p}_3+mc)\gamma^\lambda(\not{p}_4+mc)\Big].\end{aligned}$$

$$\begin{aligned}\langle|\mathcal{M}_2|^2\rangle &= \frac{B^2}{4}Q_{\mu\lambda}Q_{\nu\kappa}\mathrm{Tr}\Big[\gamma^\mu(\not{p}_1+\not{p}_2+mc)\gamma^\nu(\not{p}_1+mc)\gamma^\kappa\\ &\qquad\times(\not{p}_1+\not{p}_2+mc)\gamma^\lambda(\not{p}_4+mc)\Big].\end{aligned}$$

$$\langle \mathcal{M}_1 \mathcal{M}_2^* \rangle = \frac{AB}{4} Q_{\mu\lambda} Q_{\nu\kappa} \mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 - \not{p}_3 + mc)\gamma^\nu (\not{p}_1 + mc)\gamma^\lambda \times (\not{p}_1 + \not{p}_2 + mc)\gamma^\kappa (\not{p}_4 + mc)\Big].$$

$$\langle \mathcal{M}_2 \mathcal{M}_1^* \rangle = \frac{AB}{4} Q_{\mu\lambda} Q_{\nu\kappa} \mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 + \not{p}_2 + mc)\gamma^\nu (\not{p}_1 + mc)\gamma^\lambda \times (\not{p}_1 - \not{p}_3 + mc)\gamma^\kappa (\not{p}_4 + mc)\Big].$$

Conclusion:

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} Q_{\mu\lambda} Q_{\nu\kappa} \Big\{ & A^2 \mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 - \not{p}_3 + mc)\gamma^\nu (\not{p}_1 + mc)\gamma^\kappa \\ & \times (\not{p}_1 - \not{p}_3 + mc)\gamma^\lambda (\not{p}_4 + mc)\Big] \\ & + B^2 \mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 + \not{p}_2 + mc)\gamma^\nu (\not{p}_1 + mc)\gamma^\kappa (\not{p}_1 + \not{p}_2 + mc)\gamma^\lambda (\not{p}_4 + mc)\Big] \\ & + AB \mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 - \not{p}_3 + mc)\gamma^\nu (\not{p}_1 + mc)\gamma^\lambda (\not{p}_1 + \not{p}_2 + mc)\gamma^\kappa (\not{p}_4 + mc)\Big] \\ & + AB \mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 + \not{p}_2 + mc)\gamma^\nu (\not{p}_1 + mc)\gamma^\lambda (\not{p}_1 - \not{p}_3 + mc)\gamma^\kappa (\not{p}_4 + mc)\Big] \Big\}\end{aligned}$$

where A, B, and $Q_{\mu\nu}$ are defined above.

Problem 7.31

(a) $\mathrm{Tr}\, A \equiv \sum A_{ii}$, so

$$\begin{aligned}&(\mathbf{1}) \quad \mathrm{Tr}\,(A+B) = \sum (A+B)_{ii} = \sum A_{ii} + \sum B_{ii} = \mathrm{Tr}\, A + \mathrm{Tr}\, B. \\ &(\mathbf{2}) \quad \mathrm{Tr}\,(\alpha A) = \sum (\alpha A)_{ii} = \sum \alpha (A_{ii}) = \alpha \mathrm{Tr}\, A. \\ &(\mathbf{3}) \quad \mathrm{Tr}\,(AB) = \sum (AB)_{ii} = \sum_i \sum_j (A_{ij} B_{ji}) = \sum_i \sum_j B_{ji} A_{ij} \\ &\qquad = \sum_j (BA)_{jj} = \mathrm{Tr}\,(BA).\end{aligned}$$

(b)

$$\begin{aligned} g_{\mu\nu} g^{\mu\nu} &= g_{00} g^{00} + g_{01} g^{01} + \cdots + g_{33} g^{33}, \text{ but } g^{\mu\nu} = 0 \text{ unless } \mu = \nu \\ &= g_{00} g^{00} + g_{11} g^{11} + g_{22} g^{22} + g_{33} g^{33} \\ &= (1)(1) + (-1)(-1) + (-1)(-1) + (-1)(-1) = 4. \checkmark \end{aligned}$$

(c)

$$\not{a}\not{b}+\not{b}\not{a}=(a_\mu\gamma^\mu)(b_\nu\gamma^\nu)+(b_\nu\gamma^\nu)(a_\mu\gamma^\mu)=a_\mu b_\nu(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)$$
$$=a_\mu b_\nu(2g^{\mu\nu})=2a_\mu b^\mu=2a\cdot b\ \checkmark$$

Problem 7.32

(a)

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$$
$$\Longrightarrow g_{\mu\nu}\gamma^\mu\gamma^\nu+g_{\mu\nu}\gamma^\nu\gamma^\mu=2g_{\mu\nu}g^{\mu\nu}=2\cdot 4=8$$
$$\gamma_\mu\gamma^\mu+\gamma_\mu\gamma^\mu=2\gamma_\mu\gamma^\mu=8\Longrightarrow\gamma_\mu\gamma^\mu=4.\ \checkmark$$

$$\gamma_\mu\gamma^\nu\gamma^\mu=\gamma_\mu(2g^{\mu\nu}-\gamma^\mu\gamma^\nu)=2\gamma_\mu g^{\mu\nu}-\gamma_\mu\gamma^\mu\gamma^\nu$$
$$=2\gamma^\nu-4\gamma^\nu=-2\gamma^\nu.\ \checkmark$$

$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu=\gamma_\mu\gamma^\nu(2g^{\lambda\mu}-\gamma^\mu\gamma^\lambda)=2\underbrace{g^{\lambda\mu}\gamma_\mu}_{\gamma^\lambda}\gamma^\nu-\underbrace{(\gamma_\mu\gamma^\nu\gamma^\mu)}_{-2\gamma^\nu}\gamma^\lambda$$
$$=2(\gamma^\lambda\gamma^\nu+\gamma^\nu\gamma^\lambda)=2(2g^{\lambda\nu})=4g^{\nu\lambda}.\ \checkmark$$

$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu$$
$$=\gamma_\mu\gamma^\nu\gamma^\lambda(2g^{\sigma\mu}-\gamma^\mu\gamma^\sigma)=2\underbrace{g^{\sigma\mu}\gamma_\mu}_{\gamma^\sigma}\gamma^\nu\gamma^\lambda-\underbrace{\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu}_{4g^{\nu\lambda}}\gamma^\sigma$$
$$=2\gamma^\sigma\gamma^\nu\gamma^\lambda-4g^{\nu\lambda}\gamma^\sigma=2\gamma^\sigma\left(2g^{\nu\lambda}-\gamma^\lambda\gamma^\nu\right)-4g^{\nu\lambda}\gamma^\sigma$$
$$=4g^{\nu\lambda}\gamma^\sigma-2\gamma^\sigma\gamma^\lambda\gamma^\nu-4g^{\nu\lambda}\gamma^\sigma=-2\gamma^\sigma\gamma^\lambda\gamma^\nu.\ \checkmark$$

(b)

$$\gamma_\mu\not{a}\gamma^\mu=\gamma_\mu(a_\nu\gamma^\nu)\gamma^\mu=a_\nu(\gamma_\mu\gamma^\nu\gamma^\mu)=a_\nu(-2\gamma^\nu)=-2(a_\nu\gamma^\nu)=-2\not{a}\ \checkmark$$

$$\gamma_\mu\not{a}\not{b}\gamma^\mu=\gamma_\mu(a_\nu\gamma^\nu)(b_\lambda\gamma^\lambda)\gamma^\mu=a_\nu b_\lambda(\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu)=a_\nu b_\lambda(4g^{\nu\lambda})=4a\cdot b\ \checkmark$$

$$\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu=\gamma_\mu(a_\nu\gamma^\nu)(b_\lambda\gamma^\lambda)(c_\sigma\gamma^\sigma)\gamma^\mu=a_\nu b_\lambda c_\sigma(\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu)$$
$$=a_\nu b_\lambda c_\sigma(-2\gamma^\sigma\gamma^\lambda\gamma^\nu)=-2(c_\sigma\gamma^\sigma)(b_\lambda\gamma^\lambda)(a_\nu\gamma^\nu)=-2\not{c}\not{b}\not{a}\ \checkmark$$

Problem 7.33

(a) (10) The trace of the product of an odd number of gamma matrices is zero.
Proof: Insert $\gamma^5\gamma^5 = 1$: $\text{Tr}\,(\gamma^\mu\gamma^\nu\cdots\gamma^\sigma) = \text{Tr}\,(\gamma^\mu\gamma^\nu\cdots\gamma^\sigma\gamma^5\gamma^5)$. But γ^5 anticommutes with every γ^μ, so, pulling one γ^5 through to the left we pick up a factor of $(-1)^n$, if there are n γ's:

$$\begin{aligned}\text{Tr}\,(\gamma^\mu\gamma^\nu\cdots\gamma^\sigma) &= (-1)^n\text{Tr}\,(\gamma^5\gamma^\mu\gamma^\nu\cdots\gamma^\sigma\gamma^5)\\ &= (-1)^n\text{Tr}\,(\gamma^\mu\gamma^\nu\cdots\gamma^\sigma\gamma^5\gamma^5)\end{aligned}$$

[by $\text{Tr}\,(AB) = \text{Tr}\,(BA)$]. So $\text{Tr}\,(\gamma^\mu\gamma^\nu\cdots\gamma^\sigma) = (-1)^n\text{Tr}\,(\gamma^\mu\gamma^\nu\cdots\gamma^\sigma)$. So if n is *odd*, the trace is *zero*. ✓

(11)

$$\text{Tr}\,(1) = \text{Tr}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = 1+1+1+1 = 4. \checkmark$$

(12)

$$\begin{aligned}\text{Tr}\,(\gamma^\mu\gamma^\nu) &= \frac{1}{2}\left[\text{Tr}\,(\gamma^\mu\gamma^\nu) + \text{Tr}\,(\gamma^\nu\gamma^\mu)\right]\\ &\qquad\text{(since they're equal, by trace rule 3)}\\ &= \frac{1}{2}\text{Tr}\,(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu) = \frac{1}{2}\text{Tr}\,(2g^{\mu\nu}) = g^{\mu\nu}\text{Tr}\,(1) = 4g^{\mu\nu}. \checkmark\end{aligned}$$

(13)

$$\begin{aligned}&\text{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)\\ &\quad= \text{Tr}\,\left[(2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^\lambda\gamma^\sigma\right]\\ &\quad= 2g^{\mu\nu}\text{Tr}\,(\gamma^\lambda\gamma^\sigma) - \text{Tr}\,(\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\sigma)\\ &\quad= 8g^{\mu\nu}g^{\lambda\sigma} - \text{Tr}\,\left[\gamma^\nu(2g^{\mu\lambda}-\gamma^\lambda\gamma^\mu)\gamma^\sigma\right]\\ &\quad= 8g^{\mu\nu}g^{\lambda\sigma} - 2g^{\mu\lambda}\text{Tr}\,(\gamma^\nu\gamma^\sigma) + \text{Tr}\,(\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\sigma)\\ &\quad= 8g^{\mu\nu}g^{\lambda\sigma} - 8g^{\mu\lambda}g^{\nu\sigma} + \text{Tr}\,\left[\gamma^\nu\gamma^\lambda(2g^{\mu\sigma}-\gamma^\sigma\gamma^\mu)\right]\\ &\quad= 8g^{\mu\nu}g^{\lambda\sigma} - 8g^{\mu\lambda}g^{\nu\sigma} + 2g^{\mu\sigma}\text{Tr}\,(\gamma^\nu\gamma^\lambda) - \text{Tr}\,(\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu).\end{aligned}$$

But the last term is equal to $-\text{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)$ by trace theorem 3, so

$$2\text{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 8(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\mu\lambda}), \text{ or}$$
$$\text{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\mu\lambda}) \checkmark$$

(b)

$$\text{Tr}\,(\not{a}\not{b}) = a_\mu b_\nu \text{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}a_\mu b_\nu = 4a\cdot b. \checkmark$$

$$\begin{aligned}\text{Tr}\,(\not{a}\not{b}\not{c}\not{d}) &= a_\mu b_\nu c_\lambda d_\sigma \text{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)\\ &= 4a_\mu b_\nu c_\lambda d_\sigma(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})\\ &= 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right] \checkmark\end{aligned}$$

Problem 7.34

(a)

Theorem 14 $\text{Tr}(\gamma^5) = \text{Tr}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 0. \quad \checkmark$

Theorem 15 $\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu)$ is antisymmetric in $\mu \leftrightarrow \nu$. *Proof:* from the fundamental anticommutator,

$$\begin{aligned}\text{Tr}(\gamma^5\{\gamma^\mu,\gamma^\nu\}) &= \text{Tr}\left[\gamma^5(2g^{\mu\nu})\right] = 2g^{\mu\nu}\text{Tr}(\gamma^5) = 0\\ &= \text{Tr}(\gamma^5\gamma^\mu\gamma^\nu) + \text{Tr}(\gamma^5\gamma^\nu\gamma^\mu) \quad \text{so} \quad \text{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = -\text{Tr}(\gamma^5\gamma^\nu\gamma^\mu).\end{aligned}$$

But there *is* no general antisymmetric second-rank tensor (there's $g^{\mu\nu}$, but it's *symmetric*), so $\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0. \quad \checkmark$

[In case you don't like that argument, note that if $\mu = \nu$, then $\gamma^\mu\gamma^\nu = \pm 1$, and we are left with $\text{Tr}(\gamma^5)$, which is 0. If $\mu \neq \nu$, then (after appropriate anticommutations and again reducing the square of any gamma matrix to ± 1) $\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \text{Tr}(i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^\nu)$ reduces to $\pm i$ times the trace of the product of two distinct gamma's, which is zero by Theorem 12.]

Theorem 16 By the same argument, $\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)$ is completely antisymmetric under interchange of any two indices, but there *does* exist a general antisymmetric *fourth*-rank tensor, namely the Levi-Civita symbol, $\epsilon^{\mu\nu\lambda\sigma}$ (Eq. 7.127). All we have to do is calculate *one* permutation of the indices, and the others are all determined by invoking antisymmetry:

$$\text{Tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) = \text{Tr}[\gamma^5(-i\gamma^5)] = -i\text{Tr}(1) = -4i$$

Since $\epsilon^{0123} \equiv -1$, it follows that $\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}. \quad \checkmark$

(b)

Theorem 15' $\mathrm{Tr}(\gamma^5 \not{a}\not{b}) = a_\mu b_\nu \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0.$ ✓

Theorem 16' $\mathrm{Tr}(\gamma^5 \not{a}\not{b}\not{c}\not{d}) = a_\mu b_\nu c_\lambda d_\sigma \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i a_\mu b_\nu c_\lambda d_\sigma \epsilon^{\mu\nu\lambda\sigma}.$ ✓

Problem 7.35

(a) Since $\epsilon^{\mu\nu\lambda\sigma}$ is zero unless the four indices are all different, $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\tau}$ is zero unless $\sigma = \tau$: $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\tau} = A\delta^\sigma_\tau$, and it remains only to determine the number A, by working out a particular case. Pick $\sigma = \tau = 3$:

$$\epsilon^{\mu\nu\lambda 3}\epsilon_{\mu\nu\lambda 3} = A\delta^3_3 = A.$$

Here $\mu\nu\lambda$ must be some permutation of 012, and there are six such permutations (all equal), so

$$\epsilon^{\mu\nu\lambda 3}\epsilon_{\mu\nu\lambda 3} = 6\epsilon^{0123}\epsilon_{0123} = -6.$$

Evidently $A = -6$, and hence $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\tau} = -6\delta^\sigma_\tau$. ✓

(b) This time λ could be θ or τ (and σ the other), so

$$\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\theta\tau} = A\left(\delta^\lambda_\theta\delta^\sigma_\tau - \delta^\lambda_\tau\delta^\sigma_\theta\right)$$

(it's obviously antisymmetric in $\tau \leftrightarrow \theta$). To determine A, pick (for example) $\lambda = \theta = 2$, $\sigma = \tau = 3$:

$$\epsilon^{\mu\nu 23}\epsilon_{\mu\nu 23} = A\left(\delta^2_2\delta^3_3 - \delta^2_3\delta^3_2\right) = A = \epsilon^{0123}\epsilon_{0123} + \epsilon^{1023}\epsilon_{1023} = -2.$$

So

$$\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\theta\tau} = -2\left(\delta^\lambda_\theta\delta^\sigma_\tau - \delta^\lambda_\tau\delta^\sigma_\theta\right). \quad ✓$$

(c) This time there are six terms:

$$\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\phi\theta\tau} = A\left(\delta^\nu_\phi\delta^\lambda_\theta\delta^\sigma_\tau - \delta^\nu_\phi\delta^\lambda_\tau\delta^\sigma_\theta + \delta^\nu_\tau\delta^\lambda_\phi\delta^\sigma_\theta - \delta^\nu_\tau\delta^\lambda_\theta\delta^\sigma_\phi + \delta^\nu_\theta\delta^\lambda_\tau\delta^\sigma_\phi - \delta^\nu_\theta\delta^\lambda_\phi\delta^\sigma_\tau\right).$$

In particular,

$$\epsilon^{\mu 123}\epsilon_{\mu 123} = A = \epsilon^{0123}\epsilon_{0123} = -1,$$

so

$$\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\phi\theta\tau} = -\left(\delta^\nu_\phi\delta^\lambda_\theta\delta^\sigma_\tau - \delta^\nu_\phi\delta^\lambda_\tau\delta^\sigma_\theta + \delta^\nu_\tau\delta^\lambda_\phi\delta^\sigma_\theta - \delta^\nu_\tau\delta^\lambda_\theta\delta^\sigma_\phi + \delta^\nu_\theta\delta^\lambda_\tau\delta^\sigma_\phi - \delta^\nu_\theta\delta^\lambda_\phi\delta^\sigma_\tau\right).$$

(d) There are 24 terms:

$$
\begin{aligned}
&\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\omega\phi\theta\tau} \\
&= -\Big(\delta^\mu_\omega\delta^\nu_\phi\delta^\lambda_\theta\delta^\sigma_\tau - \delta^\mu_\omega\delta^\nu_\phi\delta^\lambda_\tau\delta^\sigma_\theta + \delta^\mu_\omega\delta^\nu_\tau\delta^\lambda_\phi\delta^\sigma_\theta - \delta^\mu_\omega\delta^\nu_\tau\delta^\lambda_\theta\delta^\sigma_\phi + \delta^\mu_\omega\delta^\nu_\theta\delta^\lambda_\tau\delta^\sigma_\phi - \delta^\mu_\omega\delta^\nu_\theta\delta^\lambda_\phi\delta^\sigma_\tau \\
&\quad - \delta^\mu_\phi\delta^\nu_\omega\delta^\lambda_\theta\delta^\sigma_\tau + \delta^\mu_\phi\delta^\nu_\omega\delta^\lambda_\tau\delta^\sigma_\theta - \delta^\mu_\phi\delta^\nu_\tau\delta^\lambda_\omega\delta^\sigma_\theta + \delta^\mu_\phi\delta^\nu_\tau\delta^\lambda_\theta\delta^\sigma_\omega - \delta^\mu_\phi\delta^\nu_\theta\delta^\lambda_\tau\delta^\sigma_\omega + \delta^\mu_\phi\delta^\nu_\theta\delta^\lambda_\omega\delta^\sigma_\tau \\
&\quad + \delta^\mu_\theta\delta^\nu_\omega\delta^\lambda_\phi\delta^\sigma_\tau - \delta^\mu_\theta\delta^\nu_\omega\delta^\lambda_\tau\delta^\sigma_\phi + \delta^\mu_\theta\delta^\nu_\tau\delta^\lambda_\omega\delta^\sigma_\phi - \delta^\mu_\theta\delta^\nu_\tau\delta^\lambda_\phi\delta^\sigma_\omega + \delta^\mu_\theta\delta^\nu_\phi\delta^\lambda_\tau\delta^\sigma_\omega - \delta^\mu_\theta\delta^\nu_\phi\delta^\lambda_\omega\delta^\sigma_\tau \\
&\quad - \delta^\mu_\tau\delta^\nu_\omega\delta^\lambda_\phi\delta^\sigma_\theta + \delta^\mu_\tau\delta^\nu_\omega\delta^\lambda_\theta\delta^\sigma_\phi - \delta^\mu_\tau\delta^\nu_\theta\delta^\lambda_\omega\delta^\sigma_\phi + \delta^\mu_\tau\delta^\nu_\theta\delta^\lambda_\phi\delta^\sigma_\omega - \delta^\mu_\tau\delta^\nu_\phi\delta^\lambda_\theta\delta^\sigma_\omega + \delta^\mu_\tau\delta^\nu_\phi\delta^\lambda_\omega\delta^\sigma_\theta\Big)
\end{aligned}
$$

(I got the overall factor by picking $\mu\nu\lambda\sigma = \omega\phi\theta\tau = 0123$.)

Problem 7.36

(a)

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma^\mu\gamma^\nu(1-\gamma^5)\gamma^\lambda(1+\gamma^5)\gamma_\lambda\right] \qquad\qquad [\text{but } \{\gamma^5,\gamma^\lambda\} = 0] \\
&\quad = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\lambda(1+\gamma^5)(1+\gamma^5)\gamma_\lambda\right] \qquad [\text{but } (1+\gamma^5)^2 = 2(1+\gamma^5)] \\
&\quad = 2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\lambda(1+\gamma^5)\gamma_\lambda\right] = 2\mathrm{Tr}\left[\gamma_\lambda\gamma^\mu\gamma^\nu\gamma^\lambda(1+\gamma^5)\right] \\
&\quad = 2(4g^{\mu\nu})\mathrm{Tr}(1+\gamma^5) = \boxed{32g^{\mu\nu}}.
\end{aligned}
$$

(I used 8, 11, and 14 on p. 253.)

(b) Dropping traces of odd products of gamma matrices:

$$
\begin{aligned}
&\mathrm{Tr}\left[(\not{p}+mc)(\not{q}+Mc)(\not{p}+mc)(\not{q}+Mc)\right] \\
&\quad = \mathrm{Tr}\left[(\not{p}\not{q}+mc\not{q}+Mc\not{p}+mMc^2)(\not{p}\not{q}+mc\not{q}+Mc\not{p}+mMc^2)\right] \\
&\quad = \mathrm{Tr}(\not{p}\not{q}\not{p}\not{q}) + 2mMc^2\mathrm{Tr}(\not{p}\not{q}) + 2mMc^2\mathrm{Tr}(\not{q}\not{p}) \\
&\qquad\quad + m^2c^2\mathrm{Tr}(\not{q}\not{q}) + M^2c^2\mathrm{Tr}(\not{p}\not{p}) + m^2M^2c^4\mathrm{Tr}(1) \\
&\quad = 4[(p\cdot q)(p\cdot q) - p^2q^2 + (p\cdot q)(p\cdot q)] + 4mMc^2[4(p\cdot q)] \\
&\qquad\quad + m^2c^2(4q^2) + M^2c^2(4p^2) + m^2M^2c^4(4) \\
&\quad = 8(p\cdot q)^2 - 4m^2M^2c^4 + 16mMc^2(p\cdot q) + 8m^2M^2c^4 + 4m^2M^2c^4 \\
&\quad = 8[(p\cdot q)^2 + 2mMc^2(p\cdot q) + m^2M^2c^4] = \boxed{8\left[(p\cdot q)+mMc^2\right]^2}.
\end{aligned}
$$

Problem 7.37

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathcal{M}_1^*\mathcal{M}_2 + \mathcal{M}_1\mathcal{M}_2^*,$$
$$\langle|\mathcal{M}|^2\rangle = \langle|\mathcal{M}_1|^2\rangle + \langle|\mathcal{M}_2|^2\rangle + \langle\mathcal{M}_1\mathcal{M}_2^*\rangle + \langle\mathcal{M}_1\mathcal{M}_2^*\rangle^*.$$

Here

$$\mathcal{M}_1 = -\frac{g_e^2}{(p_1-p_3)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right];$$

$$\mathcal{M}_2 = +\frac{g_e^2}{(p_1-p_4)^2}\left[\bar{u}(4)\gamma^\mu u(1)\right]\left[\bar{u}(3)\gamma_\mu u(2)\right].$$

From Eq. 7.129, with $m = M = 0$:

$$\langle|\mathcal{M}_1|^2\rangle = \frac{8g_e^4}{(p_1-p_3)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right].$$

And, switching $3 \leftrightarrow 4$,

$$\langle|\mathcal{M}_2|^2\rangle = \frac{8g_e^4}{(p_1-p_4)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4)\right].$$

$$\langle\mathcal{M}_1\mathcal{M}_2^*\rangle = \frac{1}{4}\left[\frac{-g_e^4}{(p_1-p_3)^2(p_1-p_4)^2}\right] \times \underbrace{\sum_{\text{spins}}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\underbrace{\left[\bar{u}(4)\gamma^\nu u(1)\right]^*}_{[\bar{u}(1)\gamma^\nu u(4)]}\underbrace{\left[\bar{u}(3)\gamma_\nu u(2)\right]^*}_{[\bar{u}(2)\gamma_\nu u(3)]}}_{\bigstar}$$

where

$$\begin{aligned}\bigstar &= \sum_{s_3}\bar{u}(3)\gamma^\mu\underbrace{\left(\sum_{s_1}u(1)\bar{u}(1)\right)}_{(\not p_1+mc)}\gamma^\nu\underbrace{\left(\sum_{s_4}u(4)\bar{u}(4)\right)}_{(\not p_4+mc)}\gamma_\mu\underbrace{\left(\sum_{s_2}u(2)\bar{u}(2)\right)}_{(\not p_2+mc)}\gamma_\nu u(3)\\ &= \sum_{s_3}\bar{u}(3)\underbrace{(\gamma^\mu\not p_1\gamma^\nu\not p_4\gamma_\mu\not p_2\gamma_\nu)}_{Q}u(3) = \sum_{i,j}\sum_{s_3}\bar{u}(3)_iQ_{ij}u(3)_j\\ &= \sum_{i,j}Q_{ij}\underbrace{\left(\sum_{s_3}u(3)\bar{u}(3)\right)_{ji}}_{(\not p_3+mc)_{ji}} = \sum_i(Q\not p_3)_{ii} = \text{Tr}\,(Q\not p_3)\\ &= \text{Tr}\,(\gamma^\mu\not p_1\gamma^\nu\not p_4\gamma_\mu\not p_2\gamma_\nu\not p_3)\end{aligned}$$

$$\langle \mathcal{M}_1 \mathcal{M}_2^* \rangle = \left[\frac{-g_e^4}{4(p_1 - p_3)^2 (p_1 - p_4)^2} \right] \text{Tr}\, (\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_3).$$

But $\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4 \gamma_\mu = -2\not{p}_4 \gamma^\nu \not{p}_1$ (theorem 9 on p. 253), so

$$\begin{aligned}
\text{Tr}\, (\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_3) &= -2\text{Tr}\, (\not{p}_4 \gamma^\nu \not{p}_1 \not{p}_2 \gamma_\nu \not{p}_3), \\
[\text{and } \gamma^\nu \not{p}_1 \not{p}_2 \gamma_\nu = 4p_1 \cdot p_2, \text{ (theorem 8) }] \quad & \\
&= -8(p_1 \cdot p_2) \text{Tr}\, (\not{p}_4 \not{p}_3) \\
&= -8(p_1 \cdot p_2)(4)(p_3 \cdot p_4) \quad \text{(theorem 12')}.
\end{aligned}$$

Thus

$$\langle \mathcal{M}_1 \mathcal{M}_2^* \rangle = \frac{8g_e^4}{(p_1 - p_3)^2 (p_1 - p_4)^2} (p_1 \cdot p_2)(p_3 \cdot p_4) = \langle \mathcal{M}_1 \mathcal{M}_2^* \rangle^*$$

(since it is clearly *real*).

Now,

$$\begin{aligned}
(p_1 - p_3)^2 &= p_1^2 + p_3^2 - 2p_1 \cdot p_3 = (mc)^2 + (mc)^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3 \\
(p_1 - p_4)^2 &= p_1^2 + p_4^2 - 2p_1 \cdot p_4 = -2p_1 \cdot p_4.
\end{aligned}$$

So

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle = {} & \frac{2g_e^4}{(p_1 \cdot p_3)^2} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)] \\
& + \frac{2g_e^4}{(p_1 \cdot p_4)^2} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4)] \\
& + 2 \frac{2g_e^4}{(p_1 \cdot p_3)(p_1 \cdot p_4)} (p_1 \cdot p_2)(p_3 \cdot p_4).
\end{aligned}$$

But

$$\begin{aligned}
& (p_1 + p_2) = (p_3 + p_4) \\
& \Longrightarrow p_1^2 + p_2^2 + 2p_1 \cdot p_2 = p_3^2 + p_4^2 + 2p_3 \cdot p_4 \\
& \Longrightarrow p_1 \cdot p_2 = p_3 \cdot p_4 \quad (\text{since } p_i^2 = m^2c^2 = 0).
\end{aligned}$$

Likewise

$$\begin{aligned}
& p_1 - p_3 = p_4 - p_2 \\
& \Longrightarrow p_1^2 + p_3^2 - 2p_1 \cdot p_3 = p_4^2 + p_2^2 - 2p_2 \cdot p_4 \\
& \Longrightarrow p_1 \cdot p_3 = p_2 \cdot p_4;
\end{aligned}$$

and

$$p_1 - p_4 = p_3 - p_2 \Longrightarrow p_1 \cdot p_4 = p_2 \cdot p_3.$$

So

$$\langle|\mathcal{M}|^2\rangle = \frac{2g_e^4}{(p_1\cdot p_3)^2(p_1\cdot p_4)^2}\Big\{(p_1\cdot p_4)^4 + (p_1\cdot p_3)^4$$

$$+ \underbrace{(p_1\cdot p_4)^2(p_1\cdot p_2)^2 + (p_1\cdot p_3)^2(p_1\cdot p_2)^2 + 2(p_1\cdot p_3)(p_1\cdot p_4)(p_1\cdot p_2)^2}_{\blacklozenge}\Big\}$$

where

$$\blacklozenge = (p_1\cdot p_2)^2\left[\underbrace{(p_1\cdot p_4)^2 + (p_1\cdot p_3)^2 + 2(p_1\cdot p_3)(p_1\cdot p_4)}_{(p_1\cdot p_4 + p_1\cdot p_3)^2}\right]$$

where, in turn,

$$\begin{aligned}(p_1\cdot p_4 + p_1\cdot p_3)^2 &= [p_1\cdot(p_3+p_4)]^2 = [p_1\cdot(p_1+p_2)]^2 \\ &= \left[p_1^2 + (p_1\cdot p_2)\right]^2 = \left[mc^2 + (p_1\cdot p_2)\right]^2 = (p_1\cdot p_2)^2\end{aligned}$$

$$\langle|\mathcal{M}|^2\rangle = \boxed{\frac{2g_e^4}{(p_1\cdot p_3)^2(p_1\cdot p_4)^2}\left[(p_1\cdot p_2)^4 + (p_1\cdot p_3)^4 + (p_1\cdot p_4)^4\right].}$$

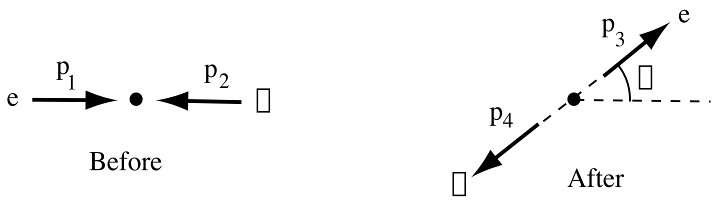
Problem 7.38

(a)

$$\langle|\mathcal{M}|^2\rangle = \frac{8g_e^4}{(p_1-p_3)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]$$

$$(p_1-p_3)^2 = p_1^2 + p_3^2 - 2p_1\cdot p_3 = -2p_1\cdot p_3$$

$$\langle|\mathcal{M}|^2\rangle = \frac{2g_e^4}{(p_1\cdot p_3)^2}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]$$



$$p_1 = \left(\frac{E}{c}, \mathbf{p}_i\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}_i\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}_f\right),$$
$$p_4 = \left(\frac{E}{c}, -\mathbf{p}_f\right), \quad \frac{E^2}{c^2} = \mathbf{p}_i^2 = \mathbf{p}_f^2, \quad \mathbf{p}_i \cdot \mathbf{p}_f = \frac{E^2}{c^2}\cos\theta$$

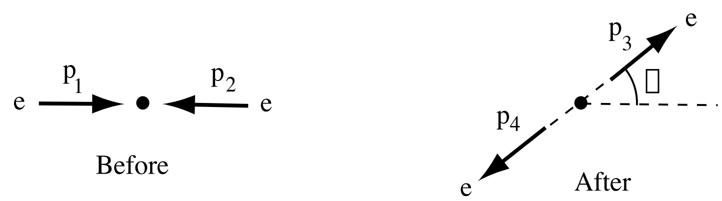
$$p_1 \cdot p_2 = \frac{E^2}{c^2} + \mathbf{p}_i^2 = 2\frac{E^2}{c^2} = p_3 \cdot p_4;$$
$$p_1 \cdot p_4 = \frac{E^2}{c^2} + \mathbf{p}_i \cdot \mathbf{p}_f = \frac{E^2}{c^2}(1+\cos\theta) = 2\frac{E^2}{c^2}\cos^2\frac{\theta}{2} = p_2 \cdot p_3;$$
$$p_1 \cdot p_3 = \frac{E^2}{c^2} - \mathbf{p}_i \cdot \mathbf{p}_f = \frac{E^2}{c^2}(1-\cos\theta) = 2\frac{E^2}{c^2}\sin^2\frac{\theta}{2}.$$

$$\langle|\mathcal{M}|^2\rangle = \frac{2g_e^4}{4(E/c)^4\sin^4\theta/2}\left(4\frac{E^4}{c^4} + 4\frac{E^4}{c^4}\cos^4\frac{\theta}{2}\right) = \boxed{2g_e^4\left(\frac{1+\cos^4\theta/2}{\sin^4\theta/2}\right)}.$$

(b) Put result of part (a) into Eq. 6.47, with $S = 1$, $|\mathbf{p}_i| = |\mathbf{p}_f|$, $E_1 + E_2 = 2E$:

$$\frac{d\sigma}{d\Omega} = \boxed{\left(\frac{\hbar c}{8\pi}\right)^2 \frac{g_e^4}{2E^2}\left(\frac{1+\cos^4\theta/2}{\sin^4\theta/2}\right)}.$$

Problem 7.39



(a)

$$\langle|\mathcal{M}|^2\rangle = \frac{2g_e^4}{(p_1\cdot p_3)^2(p_1\cdot p_4)^2}\left[(p_1\cdot p_2)^4 + (p_1\cdot p_3)^4 + (p_1\cdot p_4)^4\right].$$

$$p_1 = \left(\frac{E}{c}, \mathbf{p}_i\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}_i\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}_f\right),$$
$$p_4 = \left(\frac{E}{c}, -\mathbf{p}_f\right), \quad \frac{E^2}{c^2} = \mathbf{p}_i^2 = \mathbf{p}_f^2, \quad \mathbf{p}_i \cdot \mathbf{p}_f = \frac{E^2}{c^2}\cos\theta$$

$$p_1 \cdot p_2 = \frac{E^2}{c^2} + \mathbf{p}_i^2 = 2\frac{E^2}{c^2};$$
$$p_1 \cdot p_3 = \frac{E^2}{c^2} - \mathbf{p}_i \cdot \mathbf{p}_f = \frac{E^2}{c^2}(1-\cos\theta);$$
$$p_1 \cdot p_4 = \frac{E^2}{c^2} + \mathbf{p}_i \cdot \mathbf{p}_f = \frac{E^2}{c^2}(1+\cos\theta).$$

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{2g_e^4}{(E/c)^4(1-\cos\theta)^2(E/c)^4(1+\cos\theta)^2} \\
&\quad \times \left[\left(2\frac{E^2}{c^2}\right)^4 + \left(\frac{E^2}{c^2}(1-\cos\theta)\right)^4 + \left(\frac{E^2}{c^2}(1+\cos\theta)\right)^4\right] \\
&= \frac{2g_e^4}{(1-\cos^2\theta)^2}\left[16 + (1-\cos\theta)^4 + (1+\cos\theta)^4\right] \\
&= \frac{2g_e^4}{\sin^4\theta}\Big(16 + 1 - 4\cos\theta + 6\cos^2\theta - 4\cos^3\theta + \cos^4\theta \\
&\qquad + 1 + 4\cos\theta + 6\cos^2\theta + 4\cos^3\theta + \cos^4\theta\Big) \\
&= \frac{2g_e^4}{\sin^4\theta}\left(18 + 12\cos^2\theta + 2\cos^4\theta\right) \\
&= \frac{4g_e^4}{\sin^4\theta}\left(9 + 6\cos^2\theta + \cos^4\theta\right) = \frac{4g_e^4}{\sin^4\theta}\left(3+\cos^2\theta\right)^2 \\
&= \left[\frac{2g_e^2}{\sin^2\theta}(4-\sin^2\theta)\right]^2 = \boxed{\left[2g_e^2\left(1 - \frac{4}{\sin^2\theta}\right)\right]^2}.
\end{aligned}$$

(b) Put this into Eq. 6.47, with $S = \frac{1}{2}$, $|\mathbf{p}_i| = |\mathbf{p}_f|$, and $E_1 + E_2 = 2E$:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{2(2E)^2}\left[2g_e^2\left(1-\frac{4}{\sin^2\theta}\right)\right]^2 = \boxed{\frac{1}{2}\left[\frac{\hbar c g_e^2}{8\pi E}\left(1-\frac{4}{\sin^2\theta}\right)\right]^2}.$$

This is different from the result for $e + \mu \to e + \mu$ (Problem 7.38). As explained in the footnote on page 247, the *non*-relativistic results for $e + e \to e + e$ and $e + \mu \to e + \mu$ should be the same, but for the extreme relativistic regime this is no longer the case.

Problem 7.40

Restoring the complex conjugation, Eq. 7.158 reads

$$\mathcal{M}_{\text{singlet}} = -2\sqrt{2}ig_e^2\,(\boldsymbol{\epsilon}_3^* \times \boldsymbol{\epsilon}_4^*)_z\,.$$

$$|\mathcal{M}_{\text{singlet}}|^2 = 8g_e^4 \left(\boldsymbol{\epsilon}_3^* \times \boldsymbol{\epsilon}_4^*\right)_z \left(\boldsymbol{\epsilon}_3 \times \boldsymbol{\epsilon}_4\right)_z = 8g_e^4 \left(\epsilon_{3ij} \epsilon_{3i}^* \epsilon_{4j}^*\right) \left(\epsilon_{3kl} \epsilon_{3k} \epsilon_{4l}\right).$$

$$\begin{aligned}
\langle|\mathcal{M}_{\text{singlet}}|^2\rangle &= 8g_e^4 \epsilon_{3ij} \epsilon_{3kl} \sum_{s_3} \left(\epsilon_{3k} \epsilon_{3i}^*\right) \sum_{s_4} \left(\epsilon_{4l} \epsilon_{4j}^*\right) \\
&= 8g_e^4 \epsilon_{3ij} \epsilon_{3kl} \left[\delta_{ki} - \hat{p}_{3k} \hat{p}_{3i}\right] \left[\delta_{lj} - \hat{p}_{4l} \hat{p}_{4j}\right] \\
&= 8g_e^4 \Big[\epsilon_{3ij} \epsilon_{3ij} - \epsilon_{3ij} \epsilon_{3il} \hat{p}_{4l} \hat{p}_{4j} - \epsilon_{3ij} \epsilon_{3kj} \hat{p}_{3k} \hat{p}_{3i} + \left(\epsilon_{3ij} \hat{p}_{3i} \hat{p}_{4j}\right) \left(\epsilon_{3kl} \hat{p}_{3k} \hat{p}_{4l}\right)\Big] \\
&= 8g_e^4 \Big[\left(\epsilon_{312} \epsilon_{312} + \epsilon_{321} \epsilon_{321}\right) - \Big(\delta_{33} \delta_{jl} - \delta_{3l} \delta_{3j}\Big) \hat{p}_{4l} \hat{p}_{4j} \\
&\qquad - \Big(\delta_{33} \delta_{ik} - \delta_{3k} \delta_{3i}\Big) \hat{p}_{3k} \hat{p}_{3i} + \left(\hat{p}_3 \times \hat{p}_4\right)_z \left(\hat{p}_3 \times \hat{p}_4\right)_z \Big] \\
&= 8g_e^4 \Big[\cancel{2} - \cancel{\hat{p}_4 \cdot \hat{p}_4} + \hat{p}_{4z} \hat{p}_{4z} - \cancel{\hat{p}_3 \cdot \hat{p}_3} + \hat{p}_{3z} \hat{p}_{3z} + \left(\hat{p}_3 \times \hat{p}_4\right)_z \left(\hat{p}_3 \times \hat{p}_4\right)_z \Big].
\end{aligned}$$

From Eq. 7.136:

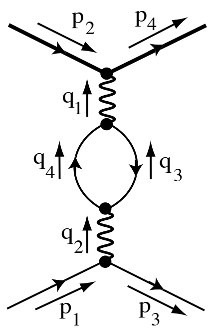
$$\hat{p}_{3z} = 1, \quad \hat{p}_{4z} = -1, \quad \hat{p}_3 \times \hat{p}_4 = 0,$$

so $\langle|\mathcal{M}_{\text{singlet}}|^2\rangle = 16g_e^4$, consistent with Eq. 7.163. ✓

This method works because, having already put the electron/positron pair in the singlet configuration, any contribution from the photon triplet is automatically zero, and hence it doesn't matter whether you include it or not.

Problem 7.41

Problem 7.42



Applying the Feynman rules,

$$\int [\bar{u}(3)\,(ig_e\gamma^\mu)\,u(1)]\,\frac{-ig_{\mu\lambda}}{q_2^2}\,[\text{LOOP}]\,\frac{-ig_{\kappa\nu}}{q_1^2}\,[\bar{u}(4)\,(ig_e\gamma^\nu)\,u(2)]$$
$$\times\,(2\pi)^4\delta^4(p_1-p_3-q_2)(2\pi)^4\delta^4(q_2-q_3-q_4)(2\pi)^4\delta^4(q_3+q_4-q_1)$$
$$\times\,(2\pi)^4\delta^4(q_1+p_2-p_4)\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}$$

where “LOOP“ stands for

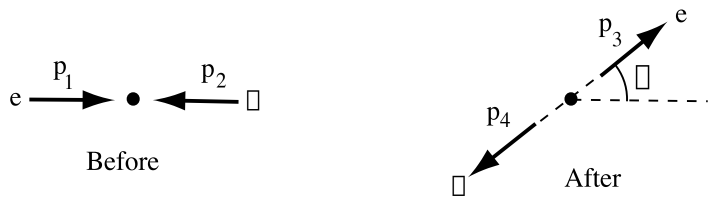
$$-\text{Tr}\left[(ig_e\gamma^\lambda)\frac{i(\not{q}_4+mc)}{q_4^2-m^2c^2}(ig_e\gamma^\kappa)\frac{i(\not{q}_3+mc)}{q_3^2-m^2c^2}\right]$$

The q_2 integral, using $\delta^4(p_1-p_3-q_2)$, sends $q_2\to p_1-p_3\equiv q$ (for short). The q_1 integral, using $\delta^4(q_1+p_2-p_4)$, sends $q_1\to p_4-p_2$, and the two remaining delta functions $\Rightarrow$ $q_3+q_4=p_1-p_3=p_4-p_2$, so q_1 is also q (of course). The q_3 integral, using $\delta^4(q-q_3-q_4)$, sends $q_3\to q-q_4$, and we erase the final delta function $(2\pi)^2\delta^4(p_1-p_3+p_2-p_4)$. The is still an integral over q_4, which (for simplicity) we rename k. Multiplying by i: $\mathcal{M}=$

$$-\frac{ig_e^4}{q^4}\,[\bar{u}(3)\gamma^\mu u(1)]\left\{\frac{d^4k}{(2\pi)^4}\frac{\text{Tr}\left[\gamma_\mu(\not{k}+mc)\gamma_\nu(\not{q}-\not{k}+mc)\right]}{[k^2-m^2c^2][(q-k)^2-m^2c^2]}\right\}[\bar{u}(4)\gamma^\nu u(2)]\ \checkmark$$

Problem 7.43

Problem 7.44



$$p_1=\left(\frac{E}{c},\mathbf{p}\right),\quad p_3=\left(\frac{E}{c},\mathbf{p}'\right),\quad |\mathbf{p}|=|\mathbf{p}'|,\quad \mathbf{p}\cdot\mathbf{p}'=\mathbf{p}^2\cos\theta,$$

$$q\equiv p_1-p_3=\left(0,\mathbf{p}-\mathbf{p}'\right),\quad q^2=-(\mathbf{p}-\mathbf{p}')^2=-\left(\mathbf{p}^2+\mathbf{p}'^2-2\mathbf{p}\cdot\mathbf{p}'\right)$$
$$-2\mathbf{p}^2(1-\cos\theta)=-4\mathbf{p}^2\sin^2\theta/2.\quad\checkmark$$

Problem 7.45

The fractional correction to the fine structure constant is

$$\frac{\Delta\alpha}{\alpha} = \frac{\alpha(0)}{3\pi} f(x), \quad \text{where} \quad x = \frac{4\mathbf{p}^2 \sin^2\theta/2}{m^2c^2}.$$

For a head-on collision $\theta = 180°$. At $v = (0.1)c$ this is nonrelativistic, so $|\mathbf{p}| = mv$:

$$x = \frac{4m^2v^2}{m^2c^2} = 4(0.1)^2 = 0.04 \ll 1,$$

so $f(x) \approx x/5 = 0.008$ (Eq. 7.185).

$$\frac{\Delta\alpha}{\alpha} = \frac{0.008}{3\pi(137)} = \boxed{6.2 \times 10^{-6}}.$$

At $E = 57.8$ GeV, by contrast, we have a highly relativistic electron, so $|\mathbf{p}| \approx E/c$, and

$$x = \frac{4E^2}{(mc^2)^2} = \frac{4(57.8 \times 10^3)^2}{(0.511)^2} = 5.12 \times 10^{10} \gg 1,$$

so (Eq. 7.185)

$$f(x) \approx \ln x = \ln\left(5.12 \times 10^{10}\right) = 24.7.$$

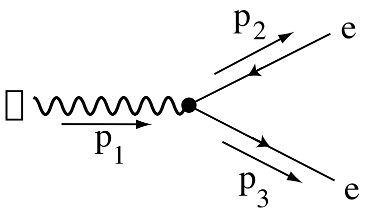
$$\frac{\Delta\alpha}{\alpha} = \frac{24.7}{3\pi(137)} = 0.0191, \quad \alpha(q^2) = \frac{1}{137}[1 + 0.0191] = 7.44 \times 10^{-3} = \frac{1}{134}$$

Problem 7.46

Problem 7.47

[See Sakurai, *Advanced Quantum Mechanics*, pp. 228-229.]

Problem 7.48



(a)

$$[\bar{u}(3)(ig_e)v(2)]\,(2\pi)^4\delta^4(p_1 - p_2 - p_3)$$

$$\mathcal{M} = -g_e\,[\bar{u}(3)v(2)]$$

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= g_e^2 \text{Tr}\,\left[(\not{p}_2 - m_2c)(\not{p}_3 + m_3c)\right] \\ &= g_e^2\left[\text{Tr}\,(\not{p}_2\not{p}_3) - (m_ec)^2\text{Tr}\,(1)\right] \\ &= g_e^2\left[4p_2\cdot p_3 - 4(m_ec)^2\right] = 4g_e^2\left[p_2\cdot p_3 - (m_ec)^2\right].\end{aligned}$$

Now

$$\begin{aligned}&p_1 = p_2 + p_3 \Longrightarrow p_1^2 = p_2^2 + p_3^2 + 2p_2\cdot p_3 \\ &\Longrightarrow (m_\gamma c)^2 = 2(m_ec)^2 + 2p_2\cdot p_3 \\ &\Longrightarrow p_2\cdot p_3 = \frac{1}{2}(m_\gamma c)^2 - (m_ec)^2.\end{aligned}$$

$$\text{So } \langle|\mathcal{M}|^2\rangle = 4g_e^2\left[\frac{1}{2}(m_\gamma c)^2 - 2(m_ec)^2\right] = \boxed{2g_e^2c^2(m_\gamma^2 - 4m_e^2)}.$$

From Eq. 6.35, with $S = 1$, we have (in the CM):

$$\Gamma = \frac{|\mathbf{p}_3|}{8\pi\hbar m_\gamma^2 c}2g_e^2c^2(m_\gamma^2 - 4m_e^2).$$

Conservation of energy $\Rightarrow m_\gamma c^2 = E_2 + E_3 = 2E_3$, so

$$E_3^2 = \frac{1}{4}m_\gamma^2c^4 = m_e^2c^4 + \mathbf{p}_3^2c^2 \Rightarrow \mathbf{p}_3^2 = \frac{1}{4}(m_\gamma^2 - 4m_e^2)c^2, |\mathbf{p}_3| = \frac{c}{2}\sqrt{m_\gamma^2 - 4m_e^2}$$

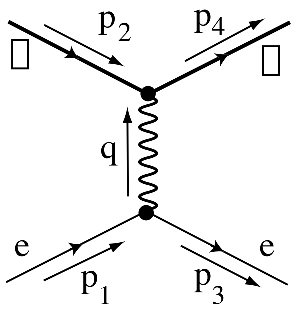
$$\Gamma = \frac{2g_e^2c^2(m_\gamma^2 - 4m_e^2)}{8\pi\hbar m_\gamma^2 c}\frac{c}{2}\sqrt{m_\gamma^2 - 4m_e^2} = \boxed{\frac{g_e^2c^2}{8\pi\hbar m_\gamma^2}(m_\gamma^2 - 4m_e^2)^{3/2}}.$$

(b)

$$\tau = \frac{1}{\Gamma} = \frac{8\pi\hbar m_\gamma^2 c^4}{\underbrace{g_e^2}_{4\pi\alpha}\left[(m_\gamma c^2)^2 - 4(m_e c^2)^2\right]^{3/2}} = \frac{2\hbar(m_\gamma c^2)^2}{\alpha\left[(m_\gamma c^2)^2 - \underbrace{4(m_e c^2)^2}_{\text{negligible}}\right]^{3/2}}$$

$$= \frac{2\hbar(m_\gamma c^2)^2}{\alpha(m_\gamma c^2)^3} = \frac{2\hbar}{\alpha(m_\gamma c^2)} = \frac{2(6.58\times 10^{-22})}{(1/137)\,(300)} = \boxed{6\times 10^{-22}\text{ seconds}}.$$

Problem 7.49



(a)

$$\int [\bar{u}(3)(ig_e)u(1)]\,\frac{-i}{q^2-(m_\gamma c)^2}\,[\bar{u}(4)(ig_e)u(2)]$$

$$\times(2\pi)^4\delta^4(p_1-p_3-q)(2\pi)^4\delta^4(p_2-p_4+q)\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = \boxed{\frac{-g_e^2}{(p_1-p_3)^2-(m_\gamma c)^2}\,[\bar{u}(3)u(1)]\,[\bar{u}(4)u(2)]}.$$

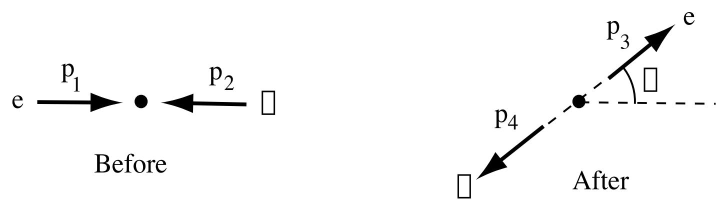
(b)

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\frac{g_e^4}{\left[(p_1-p_3)^2-(m_\gamma c)^2\right]^2}$$
$$\times\underbrace{\text{Tr}\;[(\not{p}_1+m_1c)(\not{p}_3+m_3c)]}_{\bigstar}\underbrace{\text{Tr}\;[(\not{p}_2+m_2c)(\not{p}_4+m_4c)]}_{\blacklozenge}$$

where $\bigstar = \text{Tr}\;(\not{p}_1\not{p}_3) + (m_e c)^2\text{Tr}\;(1) = 4p_1\cdot p_3 + 4(m_e c)^2$

and $\blacklozenge = \text{Tr}\,(\not p_2 \not p_4) + (m_\mu c)^2 \text{Tr}\,(1) = 4(p_2 \cdot p_4) + 4(m_\mu c)^2$.

$$\boxed{\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\frac{(2g_e)^4}{\left[(p_1 - p_3)^2 - (m_\gamma c)^2\right]^2}\left[(p_1 \cdot p_3) + (m_e c)^2\right]\left[(p_2 \cdot p_4) + (m_\mu c)^2\right]}.$$



(c)

$$p_1 = \left(\frac{E}{c}, \mathbf{p}_1\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}_1\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}_3\right), \quad p_4 = \left(\frac{E}{c}, -\mathbf{p}_3\right).$$

$$\mathbf{p}_1^2 = \mathbf{p}_3^2 = \frac{E^2}{c^2}, \quad p_1 - p_3 = (0, \mathbf{p}_1 - \mathbf{p}_3)$$

$$\begin{aligned}(p_1 - p_3)^2 &= -(\mathbf{p}_1 - \mathbf{p}_3)^2 = -\mathbf{p}_1^2 - \mathbf{p}_3^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_3 \\ &= -2\frac{E^2}{c^2} + 2\frac{E^2}{c^2}\cos\theta = -2\frac{E^2}{c^2}(1 - \cos\theta).\end{aligned}$$

$$p_1 \cdot p_3 = \frac{E^2}{c^2} - \mathbf{p}_1 \cdot \mathbf{p}_3 = \frac{E^2}{c^2}(1 - \cos\theta) = p_2 \cdot p_4.$$

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\frac{(2g_e)^4}{\left[-2(E/c)^2(1 - \cos\theta) - (m_\gamma c)^2\right]^2}\left[\frac{E^2}{c^2}(1 - \cos\theta)\right]^2.$$

From Eq. 6.47,

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\langle|\mathcal{M}|^2\rangle}{4E^2} = \boxed{\left(\frac{\hbar c g_e^2}{8\pi E}\right)^2 \left[\frac{E^2(1 - \cos\theta)}{2E^2(1 - \cos\theta) + (m_\gamma c^2)^2}\right]^2}.$$

(d)

$$\text{If } E \ll m_\gamma c^2, \quad \frac{d\sigma}{d\Omega} = \frac{1}{4}\left[\frac{\hbar c g_e^2 E}{4\pi (m_\gamma c^2)^2}\right]^2 (1 - \cos\theta)^2$$

$$\sigma = \int \frac{d\sigma}{d\Omega} \sin\theta \, d\theta \, d\phi$$
$$= \frac{2\pi}{4} \left[\frac{\hbar c g_e^2 E}{4\pi (m_\gamma c^2)^2} \right]^2 \underbrace{\int_0^\pi (1 - 2\cos\theta + \cos^2\theta) \sin\theta \, d\theta}_{\#}$$

where

$$\# = \left(-\cos\theta + \cos^2\theta - \frac{\cos^3\theta}{3} \right) \Big|_0^\pi = 2 + 0 + \frac{2}{3} = \frac{8}{3}$$

$$\sigma = \boxed{\frac{1}{3\pi} \left(\frac{\hbar g_e^2 E}{2 m_\gamma^2 c^3} \right)^3}.$$

(e) From Problem 6.8,

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_\mu c} \right)^2 \langle |\mathcal{M}|^2 \rangle.$$

From Example 7.7,

$$(p_1 - p_3)^2 = -4\mathbf{p}^2 \sin^2 \frac{\theta}{2} \ll (m_\gamma c)^2;$$
$$(p_1 \cdot p_3) = m_e^2 c^2 + 2\mathbf{p}^2 \sin^2 \frac{\theta}{2} \cong (m_e c)^2;$$
$$(p_2 \cdot p_4) = (m_\mu c)^2$$

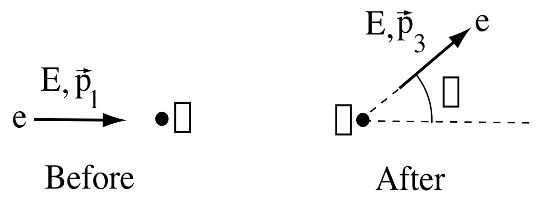
$$\langle |\mathcal{M}|^2 \rangle \cong \frac{1}{4} \frac{(2g_e)^4}{(m_\gamma c)^4} \left[2(m_e c)^2 \right] \left[2(m_\mu c)^2 \right] = \left(4 g_e^2 \frac{m_e m_\mu}{m_\gamma^2} \right)^2$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_\mu c} \right)^2 \left(4 g_e^2 \frac{m_e m_\mu}{m_\gamma^2} \right)^2 = \boxed{\left(\frac{\hbar g_e^2 m_e}{2\pi c m_\gamma^2} \right)^2}.$$

Unlike the Rutherford formula, this one is independent of E and of θ.

$$\sigma = 4\pi \frac{d\sigma}{d\Omega} = \boxed{\frac{1}{\pi} \left(\frac{\hbar g_e^2 m_e}{c m_\gamma^2} \right)^2}.$$

Alternatively, setting $m_\gamma = 0$, and treating the muon as heavy and recoilless,



$$p_1 = \left(\frac{E}{c}, \mathbf{p}\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}'\right), \quad p_2 = p_4 = (m_\mu c, \mathbf{0}); \quad (p_1 - p_3) = (0, \mathbf{p} - \mathbf{p}'),$$

$$\begin{aligned}(p_1 - p_3)^2 &= -(\mathbf{p} - \mathbf{p}')^2 = -(\mathbf{p}^2 + \mathbf{p}'^2 - 2\mathbf{p} \cdot \mathbf{p}') = -2\mathbf{p}^2(1 - \cos\theta) \\ &= -4\mathbf{p}^2 \sin^2(\theta/2);\end{aligned}$$

$$p_1 \cdot p_3 = \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p}' = \frac{E^2}{c^2} - \mathbf{p}^2 \cos\theta = m_e^2 c^2 + \mathbf{p}^2 - \mathbf{p}^2 \cos\theta \approx m_e^2 c^2,$$

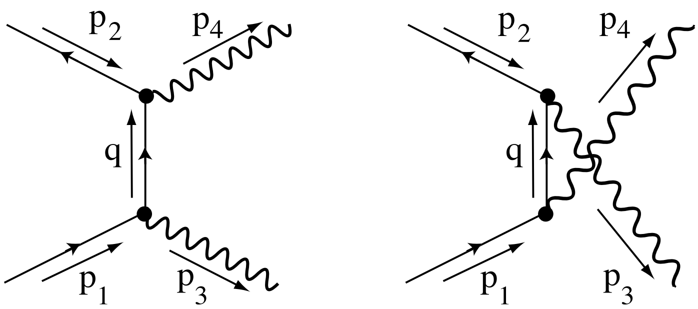
(since $|\mathbf{p}| \ll m_e c$), and $p_2 \cdot p_4 = m_\mu^2 c^2$. Inserting this into the result of part (b):

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \frac{(2g_e)^4}{(4\mathbf{p}^2 \sin^2\theta/2)^2} \left(m_e^2 c^2 + m_e^2 c^2\right) \left(m_\mu^2 c^2 + m_\mu^2 c^2\right) = \left(\frac{g_e^2 m_e m_\mu c^2}{\mathbf{p}^2 \sin^2\theta/2}\right)^2$$

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left(\frac{\hbar}{8\pi m_\mu c}\right)^2 \langle |\mathcal{M}|^2 \rangle = \left(\frac{\hbar g_e^2 m_e m_\mu c^2}{8\pi m_\mu c m_e^2 v^2 \sin^2\theta/2}\right)^2 \\ &= \left(\frac{\hbar c g_e^2}{8\pi m_e v^2 \sin^2\theta/2}\right)^2 = \left(\frac{\hbar c 4\pi e^2/\hbar c}{8\pi m_e v^2 \sin^2\theta/2}\right)^2 = \left(\frac{e^2}{2m_e v^2 \sin^2\theta/2}\right)^2,\end{aligned}$$

which is precisely the Rutherford formula (Eq. 7.132).

Problem 7.50



(a)

$$\int \left[\bar{v}(2)(ig_e) \left(\frac{i(\not{q} + m_e c)}{q^2 - (m_e c)^2} \right) ig_e u(1) \right]$$
$$\times (2\pi)^4 \delta^4(p_1 - p_3 - q)(2\pi)^4 \delta^4(p_2 + q - p_4) \frac{d^4 q}{(2\pi)^4}$$

$$\mathcal{M}_{(1)} = \frac{g_e^2}{[(p_1 - p_3)^2 - (m_e c)^2]} \left[\bar{v}(2)(\not{p}_1 - \not{p}_3 + m_e c) u(1) \right].$$

To get $\mathcal{M}_{(2)}$, switch $3 \leftrightarrow 4$; the *total*, then, is

$$\mathcal{M} = \boxed{g_e^2 \bar{v}(2) \left\{ \frac{\not{p}_1 - \not{p}_3 + m_e c}{[(p_1 - p_3)^2 - (m_e c)^2]} + \frac{\not{p}_1 - \not{p}_4 + m_e c}{[(p_1 - p_4)^2 - (m_e c)^2]} \right\} u(1)}.$$

(b) Set $m_e = m_\gamma = 0$. Note that

$$(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_e^2 c^2 + m_\gamma^2 c^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3,$$

and $(p_1 - p_4)^2 = -2p_1 \cdot p_4$. Also note that $\not{p}_1 u(1) = m_e c u(1) = 0$.

$$\mathcal{M} = \frac{g_e^2}{2} \bar{v}(2) \left[\frac{\not{p}_3}{p_1 \cdot p_3} + \frac{\not{p}_4}{p_1 \cdot p_4} \right] u(1).$$

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\left(\frac{g_e^2}{2}\right)^2 \mathrm{Tr}\left\{\left[\frac{\not p_3}{p_1\cdot p_3}+\frac{\not p_4}{p_1\cdot p_4}\right]\not p_1\left[\frac{\not p_3}{p_1\cdot p_3}+\frac{\not p_4}{p_1\cdot p_4}\right]\not p_2\right\} \\
&= \left(\frac{g_e}{2}\right)^4\left\{\frac{1}{(p_1\cdot p_3)^2}\mathrm{Tr}\,(\not p_3\not p_1\not p_3\not p_2)+\frac{1}{(p_1\cdot p_4)^2}\mathrm{Tr}\,(\not p_4\not p_1\not p_4\not p_2)\right. \\
&\qquad \left.+\frac{1}{(p_1\cdot p_3)(p_1\cdot p_4)}\left[\mathrm{Tr}\,(\not p_3\not p_1\not p_4\not p_2)+\mathrm{Tr}\,(\not p_4\not p_1\not p_3\not p_2)\right]\right\} \\
&= \left(\frac{g_e}{2}\right)^4 4\left[\frac{1}{(p_1\cdot p_3)^2}(p_3\cdot p_1 p_3\cdot p_2 - \cancel{p_3\cdot p_3} p_1\cdot p_2 + p_3\cdot p_2 p_1\cdot p_3)\right. \\
&\qquad + \frac{1}{(p_1\cdot p_4)^2}(p_4\cdot p_1 p_4\cdot p_2 - \cancel{p_4\cdot p_4} p_1\cdot p_2 + p_4\cdot p_2 p_1\cdot p_4) \\
&\qquad + \frac{1}{(p_1\cdot p_3)(p_1\cdot p_4)}\Big(p_3\cdot p_1 p_4\cdot p_2 - p_3\cdot p_4 p_1\cdot p_2 \\
&\qquad \left. + p_3\cdot p_2 p_1\cdot p_4 + p_4\cdot p_1 p_3\cdot p_2 - p_4\cdot p_3 p_1\cdot p_2 + p_4\cdot p_2 p_1\cdot p_3\Big)\right] \\
&= \left(\frac{g_e^2}{2}\right)^2\left[2\left(\frac{p_2\cdot p_3}{p_1\cdot p_3}\right)+2\left(\frac{p_2\cdot p_4}{p_1\cdot p_4}\right)+2\left(\frac{p_2\cdot p_4}{p_1\cdot p_4}\right)+2\left(\frac{p_2\cdot p_3}{p_1\cdot p_3}\right)\right. \\
&\qquad \left. - 2\frac{(p_1\cdot p_2)(p_3\cdot p_4)}{(p_1\cdot p_3)(p_1\cdot p_4)}\right] \\
&= \boxed{g_e^4\left[\frac{p_2\cdot p_3}{p_1\cdot p_3}+\frac{p_2\cdot p_4}{p_1\cdot p_4}-\frac{(p_1\cdot p_2)(p_3\cdot p_4)}{2(p_1\cdot p_3)(p_1\cdot p_4)}\right]}.
\end{aligned}$$

(c) In the CM frame,

$$p_1\cdot p_3 = p_2\cdot p_4 = \frac{E^2}{c^2}(1-\cos\theta);\quad p_1\cdot p_4 = p_2\cdot p_3 = \frac{E^2}{c^2}(1+\cos\theta);$$

$$p_1\cdot p_2 = p_3\cdot p_4 = 2\frac{E^2}{c^2}.\quad \text{So } \langle|\mathcal{M}|^2\rangle \text{ is}$$

$$\begin{aligned}
&\frac{g_e^4}{(p_1\cdot p_3)(p_1\cdot p_4)}\left[(p_2\cdot p_3)(p_1\cdot p_4)+(p_2\cdot p_4)(p_1\cdot p_3)-\tfrac{1}{2}(p_1\cdot p_2)(p_3\cdot p_4)\right] \\
&\quad = \frac{g_e^4}{(E/c)^4(1-\cos\theta)(1+\cos\theta)}\left(\frac{E}{c}\right)^4\left[(1+\cos\theta)^2+(1-\cos\theta)^2-2\right] \\
&= \frac{g_e^4}{(1-\cos^2\theta)}\left(1+2\cos\theta+\cos^2\theta+1-2\cos\theta+\cos^2\theta-2\right) = 2g_e^4\frac{\cos^2\theta}{\sin^2\theta}.
\end{aligned}$$

$$\langle|\mathcal{M}|^2\rangle = \boxed{2g_e^4\cot^2\theta}.$$

(d) Using Eq. 6.47, with $S = 1/2$,

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \left(\frac{1}{2}\right) \frac{2g_e^4 \cot^2\theta}{(2E)^2} = \boxed{\left[\frac{\hbar c g_e^2 \cot\theta}{16\pi E}\right]^2}.$$

$\boxed{\text{No}}$, σ is *not* finite. The θ integral is

$$\int_0^\pi \cot^2\theta \, \sin\theta \, d\theta = \int_0^\pi \frac{\cos^2\theta}{\sin\theta} \, d\theta;$$

near $\theta = 0$, $\cos\theta \approx 1$ while $\sin\theta \approx \theta$, so the integral goes like

$$\int_0 \frac{1}{\theta} \, d\theta \sim \ln\theta \Big|_0 \to \infty.$$

Problem 7.51

(a) Assume $\psi = i\gamma^2\psi^*$; we want to prove that $\psi' = i\gamma^2\psi'^*$. Multiply the first (on the left) by S, and note that S is real, and commutes with γ^2 (Eq. 7.53):

$$\psi' = S\psi = iS\gamma^2\psi^* = i\gamma^2 S\psi^* = i\gamma^2(S\psi)^* = i\gamma^2\psi'^*. \quad \checkmark$$

(b)

$$\psi = \begin{pmatrix} \chi_A \\ \chi_B \end{pmatrix} = i\gamma^2\psi^* = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \chi_A^* \\ \chi_B^* \end{pmatrix} = \begin{pmatrix} i\sigma_2\chi_B^* \\ -i\sigma_2\chi_A^* \end{pmatrix},$$

so

$$\chi_A = i\sigma_2\chi_B^* \quad \text{and} \quad \chi_B = -i\sigma_2\chi_A^*. \quad \checkmark$$

[These two equations are redundant. If you take the complex conjugate of the second, and multiply by σ_2, and note that σ_2 is imaginary and $(\sigma_2)^2 = 1$,

$$\sigma_2\chi_B^* = i\sigma_2\sigma_2^*\chi_A = -i(\sigma_2)^2\chi_A = -i\chi_A,$$

so $\chi_A = i\sigma_2\chi_B^*$.]

Recall that

$$\not{a} = a^0\gamma^0 - \mathbf{a}\cdot\boldsymbol{\gamma} = a^0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \mathbf{a}\cdot\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} a^0 & -\mathbf{a}\cdot\boldsymbol{\sigma} \\ \mathbf{a}\cdot\boldsymbol{\sigma} & -a^0 \end{pmatrix}.$$

For Majorana particles, the Dirac equation ($i\hbar\not{\partial}\psi - mc\,\psi = 0$) becomes,

$$i\hbar\begin{pmatrix} \partial^0 & -\boldsymbol{\sigma}\cdot\nabla \\ \boldsymbol{\sigma}\cdot\nabla & -\partial^0 \end{pmatrix}\begin{pmatrix} \chi \\ -i\sigma_2\chi^* \end{pmatrix} - mc\begin{pmatrix} \chi \\ -i\sigma_2\chi^* \end{pmatrix} = 0,$$

or (for the "upper" components)

$$i\hbar\left[\partial_0\chi + i(\boldsymbol{\sigma}\cdot\nabla)\sigma_2\chi^*\right] - mc\chi = 0. \quad \checkmark$$

[For the "lower" components we have

$$i\hbar\left[\partial_0(i\sigma_2\chi^*) + (\boldsymbol{\sigma}\cdot\nabla)\chi\right] + mc(i\sigma_2\chi^*) = 0.$$

Multiply by $-i$ and then take the complex conjugate:

$$i\hbar\left[-\sigma_2^*\partial_0\chi - i(\boldsymbol{\sigma}^*\cdot\nabla)\chi^*\right] + mc\sigma_2^*\chi = 0.$$

Use $\sigma_2^* = -\sigma_2$, and multiply by σ_2:

$$i\hbar\left[(\sigma_2)^2\partial_0\chi - i\sigma_2(\boldsymbol{\sigma}^*\cdot\nabla)\chi^*\right] - mc(\sigma_2)^2\chi = 0.$$

But $(\sigma_2)^2 = 1$, so

$$i\hbar\left[\partial_0\chi - i\sigma_2(\boldsymbol{\sigma}^*\cdot\nabla)\chi^*\right] - mc\chi = 0.$$

Evidently the two equations are consistent provided that $\sigma_2\sigma_i^* = -\sigma_i\sigma_2$. If i is 1 or 3, the Pauli matrix is real, and this is just the statement that σ_1 and σ_3 anticommutate with σ_2; if $i = 2$ the matrices commute, but we pick up an extra minus sign from the fact that σ_2 is imaginary. So it works.]

(c) The most general linear combination of plane wave solutions to the Dirac equation is

$$\psi = \begin{pmatrix} a_1 e^{-ip\cdot x/\hbar} + a_3\frac{c(p_x - ip_y)}{E+mc^2}e^{ip\cdot x/\hbar} - a_4\frac{cp_z}{E+mc^2}e^{ip\cdot x/\hbar} \\ a_2 e^{-ip\cdot x/\hbar} - a_3\frac{cp_z}{E+mc^2}e^{ip\cdot x/\hbar} - a_4\frac{c(p_x+ip_y)}{E+mc^2}e^{ip\cdot x/\hbar} \\ a_1\frac{cp_z}{E+mc^2}e^{-ip\cdot x/\hbar} + a_2\frac{c(p_x-ip_y)}{E+mc^2}e^{-ip\cdot x/\hbar} - a_4 e^{ip\cdot x/\hbar} \\ a_1\frac{c(p_x+ip_y)}{E+mc^2}e^{-ip\cdot x/\hbar} - a_2\frac{cp_z}{E+mc^2}e^{-ip\cdot x/\hbar} + a_3 e^{ip\cdot x/\hbar} \end{pmatrix}$$

For a Majorana particle,

$$\psi = \begin{pmatrix} \chi_A \\ -i\sigma_2\chi_A^* \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ -\beta^* \\ \alpha^* \end{pmatrix}$$

So the fourth row is the complex conjugate of the first row (and the third is minus the conjugate of the second, but this leads to exactly the same constraint):

$$\begin{aligned} a_1&\frac{c(p_x+ip_y)}{E+mc^2}e^{-ip\cdot x/\hbar} - a_2\frac{cp_z}{E+mc^2}e^{-ip\cdot x/\hbar} + a_3 e^{ip\cdot x/\hbar} \\ &= \left[a_1 e^{-ip\cdot x/\hbar} + a_3\frac{c(p_x-ip_y)}{E+mc^2}e^{ip\cdot x/\hbar} - a_4\frac{cp_z}{E+mc^2}e^{ip\cdot x/\hbar}\right]^* \\ &= a_1^* e^{ip\cdot x/\hbar} + a_3^*\frac{c(p_x+ip_y)}{E+mc^2}e^{-ip\cdot x/\hbar} - a_4^*\frac{cp_z}{E+mc^2}e^{-ip\cdot x/\hbar}, \end{aligned}$$

from which it follows that $a_3 = a_1^*$ and $a_4 = a_2^*$. Picking $a_1 = 1,\ a_2 = 0$ for $\chi^{(1)}$, and $a_1 = 0,\ a_2 = 1$ for $\chi^{(2)}$, and adopting a convenient normalization:

$$\chi^{(1)} = \begin{pmatrix} (E + mc^2)e^{-ip\cdot x/\hbar} + c(p_x - ip_y)e^{ip\cdot x/\hbar} \\ -cp_z e^{ip\cdot x/\hbar} \end{pmatrix},$$

$$\chi^{(2)} = \begin{pmatrix} -cp_z e^{ip\cdot x/\hbar} \\ (E + mc^2)e^{-ip\cdot x/\hbar} - c(p_x + ip_y)e^{ip\cdot x/\hbar} \end{pmatrix}.$$

The general solution (for a given E and $\mathbf{p}$) is an arbitrary linear combination of these.

8
Electrodynamics and Chromodynamics of Quarks

Problem 8.1

(a) Referring to the figure on page 277,

$$\int [\bar{v}(2)(ig_e\gamma^\mu)u(1)]\,\frac{-ig_{\mu\nu}}{q^2}\,[\bar{u}(3)(-ig_eQ\gamma^\nu)v(4)]$$
$$\times\,(2\pi)^4\delta^4(p_1+p_2-q)(2\pi)^4\delta^4(q-p_3-p_4)\,\frac{d^4q}{(2\pi)^4}$$
$$= -i\frac{Qg_e^2}{(p_1+p_2)^2}\,[\bar{v}(2)\gamma^\mu u(1)]\left[\bar{u}(3)\gamma_\mu)v(4)\right].$$

$$\mathcal{M} = \frac{Qg_e^2}{(p_1+p_2)^2}\,[\bar{v}(2)\gamma^\mu u(1)]\left[\bar{u}(3)\gamma_\mu)v(4)\right]. \quad ✓$$

(b) Using Casimir's trick,

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{\text{spins}}\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2$$
$$\times\,[\bar{v}(2)\gamma^\mu u(1)]\left[\bar{u}(3)\gamma_\mu v(4)\right][\bar{v}(4)\gamma_\nu u(3)]\,[\bar{u}(1)\gamma^\nu v(2)]$$
$$= \frac{1}{4}\sum_{\text{spins}}\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2$$
$$\times\,\mathrm{Tr}\,[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_2-mc)]\,\mathrm{Tr}\left[\gamma_\mu(\not{p}_4-Mc)\gamma_\nu(\not{p}_3+Mc)\right]. ✓$$

(c) Dropping traces of odd products of gamma matrices,

$$\mathrm{Tr}\,[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_2-mc)] = \mathrm{Tr}\,(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) - (mc)^2\mathrm{Tr}\,(\gamma^\mu\gamma^\nu)$$
$$= p_{1\kappa}p_{2\lambda}\mathrm{Tr}\left(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda\right) - (mc)^2\mathrm{Tr}\,(\gamma^\mu\gamma^\nu)$$
$$= p_{1\kappa}p_{2\lambda}4\left(g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\nu}g^{\kappa\lambda} + g^{\mu\lambda}g^{\nu\kappa}\right) - (mc)^2 4g^{\mu\nu}$$
$$= 4\left[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_1^\nu p_2^\mu - g^{\mu\nu}(mc)^2\right].$$

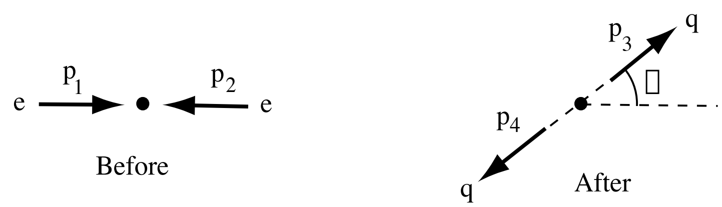
Likewise,

$$\begin{aligned}&\mathrm{Tr}\left[\gamma_\mu(\not{p}_4 - Mc)\gamma_\nu(\not{p}_3 + Mc)\right]\\&\quad = 4\left\{p_{4_\mu}p_{3_\nu} + p_{4_\nu}p_{3_\mu} - g_{\mu\nu}\left[(p_3\cdot p_4) + (Mc)^2\right]\right\}.\end{aligned}$$

So

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2 4\left\{p_1{}^\mu p_2{}^\nu + p_2{}^\nu p_1{}^\mu - g^{\mu\nu}\left[(p_1\cdot p_2) + (mc)^2\right]\right\}\\
&\qquad\times 4\left\{p_{3_\mu}p_{4_\nu} + p_{4_\mu}p_{3_\nu} - g_{\mu\nu}\left[(p_3\cdot p_4) + (Mc)^2\right]\right\}\\
&= 4\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2 \left[2(p_1\cdot p_3)(p_2\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)\right.\\
&\quad -2\,(p_1\cdot p_2)(p_3\cdot p_4) - 2(p_1\cdot p_2)(Mc)^2 - 2(p_1\cdot p_2)(p_3\cdot p_4)\\
&\quad -2\,(p_3\cdot p_4)(mc)^2 + 4(p_1\cdot p_2)(p_3\cdot p_4) + 4(p_1\cdot p_2)(Mc)^2\\
&\quad \left.+ 4(p_3\cdot p_4)(mc)^2 + 4(mMc^2)^2\right]\\
&= 8\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2 \left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right.\\
&\quad \left.+(p_1\cdot p_2)(Mc)^2 + (p_3\cdot p_4)(mc)^2 + 2(mMc^2)^2\right]. \quad \checkmark\end{aligned}$$

(d)



$$p_1 = \left(\frac{E}{c}, \mathbf{p}\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}'\right), \quad p_4 = \left(\frac{E}{c}, -\mathbf{p}'\right);$$

$$(p_1+p_2)^2 = \left(\frac{2E}{c}\right)^2, \quad p_1\cdot p_2 = \left(\frac{E}{c}\right)^2 + \mathbf{p}^2, \quad p_3\cdot p_4 = \left(\frac{E}{c}\right)^2 + \mathbf{p}'^2;$$

$$p_1\cdot p_3 = p_2\cdot p_4 = \left(\frac{E}{c}\right)^2 - \mathbf{p}\cdot\mathbf{p}', \quad p_1\cdot p_4 = p_2\cdot p_3 = \left(\frac{E}{c}\right)^2 + \mathbf{p}\cdot\mathbf{p}'.$$

$$\mathbf{p}^2 = \left(\frac{E}{c}\right)^2 - (mc)^2, \quad \mathbf{p}'^2 = \left(\frac{E}{c}\right)^2 - (Mc)^2, \quad \mathbf{p}\cdot\mathbf{p}' = |\mathbf{p}||\mathbf{p}'|\cos\theta.$$

$$p_1\cdot p_2 = 2\left(\frac{E}{c}\right)^2 - (mc)^2, \quad p_3\cdot p_4 = 2\left(\frac{E}{c}\right)^2 - (Mc)^2;$$

$$(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) = 2\left[\left(\frac{E}{c}\right)^4 + (\mathbf{p}\cdot\mathbf{p}')^2\right]$$
$$= 2\left\{\left(\frac{E}{c}\right)^4 + \left[\left(\frac{E}{c}\right)^2 - (mc)^2\right]\left[\left(\frac{E}{c}\right)^2 - (Mc)^2\right]\cos^2\theta\right\}$$

$$\begin{aligned}
&\langle|\mathcal{M}|^2\rangle \\
&= 8\left[\frac{Qg_e^2}{4(E/c)^2}\right]^2\left(2\left\{\left(\frac{E}{c}\right)^4 + \left[\left(\frac{E}{c}\right)^2 - (mc)^2\right]\left[\left(\frac{E}{c}\right)^2 - (Mc)^2\right]\cos^2\theta\right\}\right. \\
&\quad \left. + \left[2\left(\frac{E}{c}\right)^2 - (mc)^2\right](Mc)^2 + \left[2\left(\frac{E}{c}\right)^2 - (Mc)^2\right](mc)^2 + 2(mMc^2)^2\right) \\
&= \left[\frac{Qg_e^2}{(E/c)^2}\right]^2\left\{\left(\frac{E}{c}\right)^4 + (ME)^2 + (mE)^2\right. \\
&\qquad \left. + \left[\left(\frac{E}{c}\right)^2 - (mc)^2\right]\left[\left(\frac{E}{c}\right)^2 - (Mc)^2\right]\cos^2\theta\right\} \\
&= Q^2 g_e^4\left\{1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2\right. \\
&\qquad \left. + \left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right\}. \quad \checkmark
\end{aligned}$$

Problem 8.2

From Eq. 6.47 and Problem 8.1:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{(Qg_e^2)^2}{(2E)^2}&\left\{1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2\right. \\
&\left.\left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right\}\frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}
\end{aligned}$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\sin\theta\, d\theta\, d\phi, \ \int_0^{2\pi} d\phi = 2\pi, \ \int_0^{\pi}\sin\theta\, d\theta = 2, \ \int_0^{\pi}\cos^2\theta\sin\theta\, d\theta = \frac{2}{3}.$$

$$\sigma = \left(\frac{\hbar c Q 4\pi\alpha}{16\pi E}\right)^2 (2\pi)\frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}\left\{2\left[1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2\right]\right.$$
$$\left. + \frac{2}{3}\left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\right\}$$
$$= \frac{\pi Q^2}{12}\left(\frac{\hbar c\alpha}{E}\right)^2 \frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}\left[3 + 3\left(\frac{mc^2}{E}\right)^2 + 3\left(\frac{Mc^2}{E}\right)^2\right.$$
$$\left. + 1 - \left(\frac{mc^2}{E}\right)^2 - \left(\frac{Mc^2}{E}\right)^2 + \left(\frac{mc^2}{E}\right)^2\left(\frac{Mc^2}{E}\right)^2\right]$$
$$= \frac{\pi Q^2}{3}\left(\frac{\hbar c\alpha}{E}\right)^2 \frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}\left[1 + \frac{1}{2}\left(\frac{mc^2}{E}\right)^2 + \frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right.$$
$$\left. + \frac{1}{4}\left(\frac{mc^2}{E}\right)^2\left(\frac{Mc^2}{E}\right)^2\right]$$
$$= \frac{\pi Q^2}{3}\left(\frac{\hbar c\alpha}{E}\right)^2 \frac{\sqrt{1 - (Mc^2/E)^2}}{\sqrt{1 - (mc^2/E)^2}}\left[1 + \frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right]\left[1 + \frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right]. \checkmark$$

Problem 8.3

There's a second diagram in the elastic case, and this means that the kinematic factors do not cancel , as they do for the muons.

Problem 8.4

From Eq. 8.13,

$$q_\mu L^{\mu\nu} = 2\left\{(p_1 \cdot q)p_3^\nu + (p_3 \cdot q)p_1^\nu + q^\nu\left[(mc)^2 - (p_1 \cdot p_3)\right]\right\}.$$

$$q = p_4 - p_2 = p_1 - p_3 \Rightarrow p_1 \cdot q = p_1^2 - p_1 \cdot p_3 = (mc)^2 - p_1 \cdot p_3 = -p_3 \cdot q,$$

so

$$q_\mu L^{\mu\nu} = 2\left[(mc)^2 - p_1 \cdot p_3\right]\left[p_3^\nu - p_1^\nu + q^\nu\right] = 0.$$

In the "Breit" frame (Problem 3.24),

$$p_1 = \left(\frac{E}{c}, \quad \mathbf{p}\right), \quad p_3 = \left(\frac{E}{c}, -\mathbf{p}\right), \quad \text{so} \quad q^\mu = (0, 2\mathbf{p}),$$

and if we choose the z axis to lie along $\mathbf{p}$, then q has only a z component—that is, there exists an inertial frame in which $q^\mu = (0, 0, 0, q)$. In this frame

$q_\mu L^{\mu\nu} = qL^{3\nu} = 0$, so $L^{3\nu} = 0$, and (since $L^{\mu\nu}$ is symmetric) this means also that $L^{\nu 3} = 0$. Written out as a matrix, then,

$$L^{\mu\nu} = \begin{pmatrix} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and hence} \quad L^{\mu\nu}K_{\nu\mu} = \begin{pmatrix} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & x \\ x & x & x & x \end{pmatrix}$$

does not depend on the "x" entries (since they are all multiplied by 0; we could just as well set all these entries to zero, in which case $q_\mu K^{\mu\nu} = 0$. Moreover, since this is a covariant equation, it will then hold in *any* reference frame. (Note that $K^{\mu\nu}$ is a *symmetric* matrix.)

Problem 8.5

Using the first footnote on page 281, $p \cdot q = -q^2/2$:

$$\begin{aligned} q_\mu K^{\mu\nu} &= -K_1 q^\nu + \frac{K_2}{(Mc)^2}(p\cdot q)p^\nu + \frac{K_4}{(Mc)^2}q^2 q^\nu + \frac{K_5}{(Mc)^2}\left[(p\cdot q)q^\nu + q^2 p^\nu\right] \\ &= \frac{1}{(Mc)^2}\left\{q^\nu\left[-K_1(Mc)^2 + K_4 q^2 - \frac{K_5}{2}q^2\right] + q^2 p^\nu\left[-\frac{K_2}{2} + K_5\right]\right\} = 0. \end{aligned}$$

$$\begin{aligned} p_\nu q_\mu K^{\mu\nu} &= \left\{\frac{p\cdot q}{(Mc)^2}\left[-K_1(Mc)^2 + K_4 q^2 - \frac{K_5}{2}q^2\right] + q^2\left[-\frac{K_2}{2} + K_5\right]\right\} \\ &= \frac{q^2}{2}\left[K_1 - \frac{K_4}{(Mc)^2}q^2 + \frac{K_5}{2(Mc)^2}q^2 - K_2 + 2K_5\right] = 0. \end{aligned}$$

$$q^2 K_4 = (Mc)^2\left[K_1 - K_2 + K_5\left(2 + \frac{q^2}{2(Mc)^2}\right)\right].$$

Use this to eliminate K_4 in line 2:

$$\begin{aligned} q^\nu\left\{-K_1(Mc)^2 + (Mc)^2\left[K_1 - K_2 + K_5\left(2 + \frac{q^2}{2(Mc)^2}\right)\right] - \frac{K_5}{2}q^2\right\} \\ + q^2 p^\nu\left[-\frac{K_2}{2} + K_5\right] = 0. \end{aligned}$$

$$(Mc)^2 q^\nu(-K_2 + 2K_5) + \frac{q^2 p^\nu}{2}(-K_2 + 2K_5) = 0 \Rightarrow K_2 = 2K_5. \quad \checkmark$$

Put this into the expression for $q^2 K_4$:

$$q^2 K_4 = (Mc)^2\left[K_1 - 2K_5 + K_5\left(2 + \frac{q^2}{2(Mc)^2}\right)\right] = K_1(Mc)^2 + K_5\frac{q^2}{2}.$$

$$K_4 = K_1 \frac{(Mc)^2}{q^2} + \frac{K_5}{2}. \quad \checkmark$$

Problem 8.6

From Eq. 8.13, for a "Dirac" proton

$$K^{\mu\nu} = 2\left\{ p_2^\mu p_4^\nu + p_2^\nu p_4^\mu + g^{\mu\nu}\left[(Mc)^2 - (p_2 \cdot p_4)\right]\right\}.$$

Use $p_4 = q + p_2$, and drop the subscript on p_2:

$$\begin{aligned} K^{\mu\nu} &= 2\left\{ p^\mu(q^\nu + p^\nu) + p^\nu(q^\mu + p^\mu) + g^{\mu\nu}\left[(Mc)^2 - p\cdot(q+p)\right]\right\} \\ &= 2\left\{ p^\mu q^\nu + p^\nu q^\mu + 2p^\mu p^\nu + g^{\mu\nu}\left[(Mc)^2 + \frac{q^2}{2} - (Mc)^2\right]\right\} \end{aligned}$$

(where I used $p \cdot q = -q^2/2$ and $p^2 = (Mc)^2$), or

$$K^{\mu\nu} = 2\left(p^\mu q^\nu + p^\nu q^\mu\right) + 4p^\mu p^\nu + g^{\mu\nu} q^2.$$

Comparing the generic expression (Eq. 8.18):

$$\begin{aligned} K^{\mu\nu} &= K_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{K_2}{(Mc)^2}\left(p^\mu + \frac{1}{2}q^\mu\right)\left(p^\nu + \frac{1}{2}q^\nu\right) \\ &= \left(p^\mu q^\nu + p^\nu q^\mu\right)\frac{K_2}{2(Mc)^2} + p^\mu p^\nu \frac{K_2}{(Mc)^2} - g^{\mu\nu}K_1 + q^\mu q^\nu\left(\frac{K_1}{q^2} + \frac{K_2}{4(Mc)^2}\right), \end{aligned}$$

we read off $\boxed{K_2 = 4(Mc)^2}$ and $\boxed{K_1 = -q^2}$.

Meanwhile, from Eq. 8.24,

$$\begin{aligned} K_1 &= -q^2 G_M^2 \quad \Rightarrow \quad \boxed{G_M = 1} \\ K_2 &= (2Mc)^2 \frac{G_E^2 - [q^2/(2Mc)^2]}{1 - [q^2/(2Mc)^2]} \quad \Rightarrow \quad \boxed{G_E = 1}. \end{aligned}$$

Problem 8.7

Putting Eqs. 8.13 and 8.18 into Eq. 8.14:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle = & \frac{g_e^4}{q^4}2\left\{p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + g^{\mu\nu}\left[(mc)^2-(p_1\cdot p_3)\right]\right\} \\
& \times \left\{K_1\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+\frac{K_2}{(Mc)^2}\left(p_\mu+\tfrac{1}{2}q_\mu\right)\left(p_\nu+\tfrac{1}{2}q_\nu\right)\right\} \\
= & \frac{2g_e^4}{q^4}\left\{K_1\left[-2p_1\cdot p_3-4(mc)^2+4p_1\cdot p_3+2\frac{p_1\cdot q p_3\cdot q}{q^2}+(mc)^2-p_1\cdot p_3\right]\right. \\
& + \frac{K_2}{(Mc)^2}\left[2\left(p_1\cdot p+\tfrac{1}{2}p_1\cdot q\right)\left(p_3\cdot p+\tfrac{1}{2}p_3\cdot q\right)\right. \\
& \left.\left. + \left(p+\tfrac{1}{2}q\right)\cdot\left(p+\tfrac{1}{2}q\right)\left[(mc)^2-p_1\cdot p_3\right]\right]\right\} \\
= & \frac{2g_e^4}{q^4}\left\{K_1\left[(p_1\cdot p_3)-3(mc)^2+2\frac{(p_1\cdot q)(p_3\cdot q)}{q^2}\right]\right. \\
& + \frac{K_2}{(Mc)^2}\left[2(p_1\cdot p)(p_3\cdot p)+(p_1\cdot p)(p_3\cdot q)+(p_1\cdot q)(p_3\cdot p)\right. \\
& \left.\left. + \tfrac{1}{2}(p_1\cdot q)(p_3\cdot q)+\left(p^2+(p\cdot q)+\tfrac{1}{4}q^2\right)\left[(mc)^2-(p_1\cdot p_3)\right]\right]\right\}
\end{aligned}$$

Now

$$q = p_4 - p_2 = p_1 - p_3 \quad \Rightarrow \quad q^2 = p_1^2 + p_3^2 - 2p_1\cdot p_3 = 2\left[(mc)^2 - p_1\cdot p_3\right]$$

$$\begin{aligned}
p_1\cdot q &= p_1^2 - p_1\cdot p_3 = (mc)^2 - p_1\cdot p_3 = \tfrac{1}{2}q^2 \\
p_3\cdot q &= p_1\cdot p_3 - p_3^2 = p_1\cdot p_3 - (mc)^2 = -\tfrac{1}{2}q^2
\end{aligned}$$

$$p^2 = (Mc)^2, \quad p\cdot q = -\tfrac{1}{2}q^2$$

So the coefficient of K_1 reduces to

$$(p_1\cdot p_3) - 3(mc)^2 + \frac{2}{q^2}\left(\frac{q^2}{2}\right)\left(-\frac{q^2}{2}\right) = 2(p_1\cdot p_3) - 4(mc)^2,$$

and the coefficient of $K_2/(Mc)^2$ becomes

$$2(p_1 \cdot p)(p_3 \cdot p) + (p_1 \cdot p)\left(-\frac{q^2}{2}\right) + (p_3 \cdot p)\left(\frac{q^2}{2}\right) + \frac{1}{2}\left(\frac{q^2}{2}\right)\left(-\frac{q^2}{2}\right) + \left[(Mc)^2 - \frac{q^2}{2} + \frac{q^2}{4}\right]\frac{q^2}{2}$$

$$= 2(p_1 \cdot p)(p_3 \cdot p) + \frac{q^2}{2}\left[-(p_1 - p_3) \cdot p - \frac{q^2}{4} + (Mc)^2 - \frac{q^2}{4}\right]$$

$$= 2(p_1 \cdot p)(p_3 \cdot p) + \frac{q^2}{2}(Mc)^2.$$

Thus

$$\langle|\mathcal{M}|^2\rangle = \frac{2g_e^4}{q^4}\left\{K_1\left[2(p_1 \cdot p_3) - 4(mc)^2\right] + \frac{K_2}{(Mc)^2}\left[2(p_1 \cdot p)(p_3 \cdot p) + \frac{q^2}{2}(Mc)^2\right]\right\} \quad \checkmark$$

Problem 8.8

From the text,

$$p = (Mc, \mathbf{0}),\ p_1 = \frac{E}{c}(1, \hat{\mathbf{p}}_i),\ p_3 = \frac{E'}{c}(1, \hat{\mathbf{p}}_f),\ p_1 \cdot p = ME,\ p_3 \cdot p = ME',$$

$$p_1 \cdot p_3 = \frac{EE'}{c^2}(1 - \hat{\mathbf{p}}_i \cdot \hat{\mathbf{p}}_f) = \frac{EE'}{c^2}(1 - \cos\theta) = 2\frac{EE'}{c^2}\sin^2\theta/2.$$

From Problem 8.7 (with $m \to 0$)

$$q^2 = 2\left[(mc)^2 - (p_1 \cdot p_3)\right] = -4\frac{EE'}{c^2}\sin^2\theta/2.$$

Putting this into Eq. 8.19:

$$\langle|\mathcal{M}|^2\rangle = \frac{4g_e^4}{\left[4(EE'/c^2)\sin^2\theta/2\right]^2}\left\{K_1 2\frac{EE'}{c^2}\sin^2\theta/2 + K_2\left[\frac{EE'}{c^2} - \frac{EE'}{c^2}\sin^2\theta/2\right]\right\}$$

$$= \frac{g_e^4 c^2}{4EE'\sin^4\theta/2}\left\{2K_1\sin^2\theta/2 + K_2\cos^2\theta/2\right\}. \quad \checkmark$$

Problem 8.9

In the lab frame (see p. 282),

$$p_1 = \frac{E}{c}(1, \hat{\mathbf{p}}_i), \quad p_3 = \frac{E'}{c}(1, \hat{\mathbf{p}}_f), \quad \hat{\mathbf{p}}_i \cdot \hat{\mathbf{p}}_f = \cos\theta, \quad p_2 = (Mc, \mathbf{0}).$$

From energy-momentum conservation,

$$p_4 = p_1 + p_2 - p_3 = \left(Mc + \frac{E - E'}{c}, \frac{E\hat{\mathbf{p}}_i - E'\hat{\mathbf{p}}_f}{c} \right).$$

But $p_4^2 = (Mc)^2$, so

$$\begin{aligned}
(Mc)^2 &= \left(Mc + \frac{E - E'}{c} \right)^2 - \left(\frac{E\hat{\mathbf{p}}_i - E'\hat{\mathbf{p}}_f}{c} \right)^2 \\
&= (Mc)^2 + 2M(E - E') + \frac{(E - E')^2}{c^2} - \frac{E^2 + E'^2 - 2EE'\cos\theta}{c^2} \\
&= (Mc)^2 + 2M(E - E') - \frac{2EE'(1 - \cos\theta)}{c^2},
\end{aligned}$$

or

$$M(E - E') = \frac{EE'}{c^2}(1 - \cos\theta) = \frac{EE'}{c^2} 2\sin^2\theta/2.$$

$$ME = E'\left(M + \frac{2E}{c^2}\sin^2\theta/2 \right); \quad E' = \frac{E}{1 + (2E/Mc^2)\sin^2\theta/2}. \quad \checkmark$$

Problem 8.10

For a "Dirac" proton (Problem 8.6), $K_1 = -q^2$ and $K_2 = (2Mc)^2$, and the Rosenbluth formula (Eq. 8.23) becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{4ME\sin^2\theta/2} \right)^2 \frac{E'}{E} \left[-2q^2\sin^2\theta/2 + (2Mc)^2\cos^2\theta/2 \right].$$

If $mc^2 \ll E \ll Mc^2$, then $E' \approx E$ (Problem 8.9), and q^2 is

$$(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = 2m^2c^2 - 2\frac{EE'}{c^2}(1 - \cos\theta) \approx -4\frac{E^2}{c^2}\sin^2\theta/2,$$

so

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{\alpha\hbar}{4ME\sin^2\theta/2}\right)^2 4M^2c^2\cos^2\theta/2 = \left(\frac{\alpha\hbar c}{2E\sin^2\theta/2}\right)^2 \cos^2\theta/2.$$

But for a highly relativistic electron ($mc^2 \ll E$), $E \approx |\mathbf{p}|c$, so

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{\alpha\hbar}{2|\mathbf{p}|\sin^2\theta/2}\right)^2 \cos^2\theta/2.$$

Compare this with the Mott formula (Eq. 7.131):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{2\mathbf{p}^2\sin^2\theta/2}\right)^2 \left[(mc)^2 + \mathbf{p}^2\cos^2\theta/2\right],$$

which reduces (in the case $mc \ll |\mathbf{p}|$) to

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{\alpha\hbar}{2|\mathbf{p}|\sin^2\theta/2}\right)^2 \cos^2\theta/2. \quad \checkmark$$

Problem 8.11

[The answer is given in the text.]

Problem 8.12

From Eq. 8.28:

$$r = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad b = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad g = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

so, under the action of U,

$$r \to \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = g$$

$$b \to \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = r$$

$$g \to \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} = b.$$

Likewise, $\bar{r} \to \bar{g},\ \bar{b} \to \bar{r},\ \bar{g} \to \bar{b}$. Thus (Eq. 8.29)

$$\begin{aligned}
|3\rangle \to |3'\rangle &= (g\bar{g} - r\bar{r})/\sqrt{2} = \alpha|3\rangle + \beta|8\rangle \\
&= \alpha(r\bar{r} - b\bar{b})/\sqrt{2} + \beta(r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6} \\
&= g\bar{g}\left(-2\frac{\beta}{\sqrt{6}}\right) + r\bar{r}\left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}}\right) + b\bar{b}\left(-\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}}\right).
\end{aligned}$$

Evidently

$$\begin{aligned}
\frac{1}{\sqrt{2}} &= -2\frac{\beta}{\sqrt{6}}, \quad \boxed{\beta = -\frac{\sqrt{3}}{2}} \\
-\frac{1}{\sqrt{2}} &= \left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}}\right) = \frac{\alpha}{\sqrt{2}} - \frac{1}{2\sqrt{2}}, \quad \boxed{\alpha = -\frac{1}{2}} \\
0 &= \left(-\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}}\right) = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{6}} = 0. \quad \checkmark
\end{aligned}$$

Similarly

$$\begin{aligned}
|8\rangle \to |8'\rangle &= (g\bar{g} + r\bar{r} - 2b\bar{b})/\sqrt{6} = \gamma|3\rangle + \delta|8\rangle \\
&= \gamma(r\bar{r} - b\bar{b})/\sqrt{2} + \delta(r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6} \\
&= g\bar{g}\left(-2\frac{\delta}{\sqrt{6}}\right) + r\bar{r}\left(\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{6}}\right) + b\bar{b}\left(-\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{6}}\right).
\end{aligned}$$

Evidently

$$\begin{aligned}
\frac{1}{\sqrt{6}} &= -2\frac{\delta}{\sqrt{6}}, \quad \boxed{\delta = -\frac{1}{2}} \\
\frac{1}{\sqrt{6}} &= \left(\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{6}}\right) = \frac{\gamma}{\sqrt{2}} - \frac{1}{2\sqrt{6}}, \quad \boxed{\gamma = \frac{\sqrt{3}}{2}} \\
-\frac{2}{\sqrt{6}} &= \left(-\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{6}}\right) = -\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{6}} = -\frac{2}{\sqrt{6}}. \quad \checkmark
\end{aligned}$$

Problem 8.13

There are 64 matrix products here, and we need to calculate the trace of each one. I think it's time for Mathematica. First define the 8 lambda matrices (Eq. 8.34), then construct a matrix whose ij element is $\mathrm{Tr}(\lambda^i\lambda^j)$:

```
m1 = {{0, 1, 0}, {1, 0, 0}, {0, 0, 0}}
m2 = {{0, -I, 0}, {I, 0, 0}, {0, 0, 0}}
m3 = {{1, 0, 0}, {0, -1, 0}, {0, 0, 0}}
m4 = {{0, 0, 1}, {0, 0, 0}, {1, 0, 0}}
m5 = {{0, 0, -I}, {0, 0, 0}, {I, 0, 0}}
m6 = {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}}
m7 = {{0, 0, 0}, {0, 0, -I}, {0, I, 0}}
m8 = {{1/√3, 0, 0}, {0, 1/√3, 0}, {0, 0, -2/√3}}
M = {m1, m2, m3, m4, m5, m6, m7, m8}
MatrixForm[Table[Sum[(Part[M, i].Part[M, j])[[k, k]], {k, 3}], {i, 8}, {j, 8}]]
```

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Problem 8.14

From Problem 4.19:

$$\begin{aligned} \sigma^i \sigma^j &= \delta^{ij} + i\epsilon^{ijk}\sigma^k \\ \sigma^j \sigma^i &= \delta^{ji} + i\epsilon^{jik}\sigma^k = \delta^{ij} - i\epsilon^{ijk}\sigma^k; \end{aligned}$$

Subtracting,

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k.$$

Evidently $\boxed{f^{ijk} = \epsilon^{ijk}}$.

Problem 8.15

(a) There are 8 ways to choose α, and (since β must be different) 7 remaining ways to choose β, and 6 ways to chose γ. But the *ordering* of the three indices doesn't matter (by antisymmetry the different orderings differ at most by a minus sign), so we divide by the number of permutations on three objects (3!), and the answer is $8 \times 7 \times 6/3 \times 2 = \boxed{56}$.

(b)

$$[\lambda^1,\lambda^2] = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}i&0&0\\0&-i&0\\0&0&0\end{pmatrix} - \begin{pmatrix}-i&0&0\\0&i&0\\0&0&0\end{pmatrix} = 2i\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} = 2i\lambda^3 = 2if^{12\gamma}\lambda^\gamma.$$

Evidently $f^{12\gamma} = 0$ unless $\gamma = 3$, and $f^{123} = 1$. ✓

(c)

$$[\lambda^1,\lambda^3] = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} - \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} = 2i\begin{pmatrix}0&i&0\\-i&0&0\\0&0&0\end{pmatrix} = -2i\lambda^2 = 2if^{13\gamma}\lambda^\gamma.$$

Evidently $f^{13\gamma} = 0$ unless $\gamma = 2$, and $\boxed{f^{132} = -1}$ (so $f^{123} = 1$, which we already knew).

$$[\lambda^4,\lambda^5] = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix} - \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$= \begin{pmatrix}i&0&0\\0&0&0\\0&0&-i\end{pmatrix} - \begin{pmatrix}-i&0&0\\0&0&0\\0&0&i\end{pmatrix} = 2i\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$= 2i\left(\frac{1}{2}\lambda^3 + \frac{\sqrt{3}}{2}\lambda^8\right) = 2if^{45\gamma}\lambda^\gamma.$$

Evidently $f^{45\gamma} = 0$ unless $\gamma = 3$ or 8, and $\boxed{f^{453} = \frac{1}{2}, \quad f^{458} = \frac{\sqrt{3}}{2}}$.

Problem 8.16

(a) Here

$$c_1 = c_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad c_2 = c_4 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

From Eq. 8.47,

$$f=\frac{1}{4}\left[(0\ 1\ 0)\,\lambda^{\alpha}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[(0\ 0\ 1)\,\lambda^{\alpha}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]=\frac{1}{4}\lambda^{\alpha}_{22}\lambda^{\alpha}_{33}.$$

The only λ matrix with a nonzero entry in the 33 position is λ^8, so

$$f=\frac{1}{4}\lambda^{8}_{22}\lambda^{8}_{33}=\frac{1}{4}\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{3}}\right)=-\frac{1}{6}.\quad\checkmark$$

(b)

$$c_1=c_2=\begin{pmatrix}1\\0\\0\end{pmatrix},\text{ for the first term, }c_1=c_2=\begin{pmatrix}0\\1\\0\end{pmatrix}\text{ for the second :}$$

$$f=\frac{1}{4}\frac{1}{\sqrt{2}}\left\{\left[c_3^{\dagger}\lambda^{\alpha}\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(1\ 0\ 0)\,\lambda^{\alpha}c_4\right]-\left[c_3^{\dagger}\lambda^{\alpha}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[(0\ 1\ 0)\,\lambda^{\alpha}c_4\right]\right\}$$

The outgoing quarks are in the same color state $(r\bar{r}-b\bar{b})/\sqrt{2})$, so

$$c_3=c_4=\begin{pmatrix}1\\0\\0\end{pmatrix},\text{ for the first term, }c_3=c_4=\begin{pmatrix}0\\1\\0\end{pmatrix}\text{ for the second.}$$

There are 4 terms in all:

$$\begin{aligned}f\;=\;&\frac{1}{4}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left\{\left[(1\ 0\ 0)\,\lambda^{\alpha}\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(1\ 0\ 0)\,\lambda^{\alpha}\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\right.\\&-\left[(0\ 1\ 0)\,\lambda^{\alpha}\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(0\ 1\ 0)\,\lambda^{\alpha}\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\\&-\left[(1\ 0\ 0)\,\lambda^{\alpha}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[(0\ 1\ 0)\,\lambda^{\alpha}\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\\&\left.+\left[(0\ 1\ 0)\,\lambda^{\alpha}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[(0\ 1\ 0)\,\lambda^{\alpha}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\right\}\end{aligned}$$

$$f = \frac{1}{8}\left(\lambda^\alpha_{11}\lambda^\alpha_{11} - \lambda^\alpha_{21}\lambda^\alpha_{12} - \lambda^\alpha_{12}\lambda^\alpha_{21} + \lambda^\alpha_{22}\lambda^\alpha_{22}\right) = \frac{1}{8}\Big[\left(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11}\right)$$
$$- \left(\lambda^1_{21}\lambda^1_{12} + \lambda^2_{21}\lambda^2_{12}\right) - \left(\lambda^1_{12}\lambda^1_{21} + \lambda^2_{12}\lambda^2_{21}\right) + \left(\lambda^3_{22}\lambda^3_{22} + \lambda^8_{22}\lambda^8_{22}\right)\Big]$$
$$= \frac{1}{8}\left\{\left[(-1)(-1) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right] - [(1)(1) + (i)(-i)]\right.$$
$$\left. - [(1)(1) + (-i)(i)] + \left[(-1)(-1) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right]\right\}$$
$$= \frac{1}{8}\left(1 + \frac{1}{3} - 1 - 1 - 1 - 1 + 1 + \frac{1}{3}\right) = \frac{1}{8}\left(-\frac{4}{3}\right) = -\frac{1}{6}. \quad \checkmark$$

(c)

$$f = \frac{1}{4}\frac{1}{\sqrt{6}}\left\{\left[c_3^\dagger\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(1\ 0\ 0)\,\lambda^\alpha c_4\right] + \left[c_3^\dagger\lambda^\alpha\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[(0\ 1\ 0)\,\lambda^\alpha c_4\right]\right.$$
$$\left. - 2\left[c_3^\dagger\lambda^\alpha\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\left[(0\ 0\ 1)\,\lambda^\alpha c_4\right]\right\}$$

There are nine terms,

$$f = \frac{1}{4}\frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}\left[(\lambda^\alpha_{11}\lambda^\alpha_{11} + \lambda^\alpha_{21}\lambda^\alpha_{12} - 2\lambda^\alpha_{31}\lambda^\alpha_{13}) + (\lambda^\alpha_{12}\lambda^\alpha_{21} + \lambda^\alpha_{22}\lambda^\alpha_{22} - 2\lambda^\alpha_{32}\lambda^\alpha_{23})\right.$$
$$\left. - 2\left(\lambda^\alpha_{13}\lambda^\alpha_{31} + \lambda^\alpha_{23}\lambda^\alpha_{32} - 2\lambda^\alpha_{33}\lambda^\alpha_{33}\right)\right]$$
$$= \frac{1}{24}\left\{\left(1 + \tfrac{1}{3}\right) + (1+1) - 2(1+1) + (1+1) + \left(1 + \tfrac{1}{3}\right) - 2(1+1)\right.$$
$$\left. - 2\left[(1+1) + (1+1) - 2\left(\tfrac{4}{3}\right)\right]\right\}$$
$$= \frac{1}{24}\left(\tfrac{4}{3} + 2 - 4 + 2 + \tfrac{4}{3} - 4 - 4 - 4 + \tfrac{16}{3}\right) = \frac{1}{24}(-4) = -\frac{1}{6}. \quad \checkmark$$

Problem 8.17

Applying the Feynman rules, $\mathscr{M} =$

$$i\left[\bar{v}(2)c_2^\dagger\left(-\frac{ig_s}{2}\lambda^\alpha\gamma^\mu\right)c_1 u(1)\right]\left(\frac{-ig_{\mu\nu}\delta^{\alpha\beta}}{q^2}\right)\left[\bar{u}(3)c_3^\dagger\left(-\frac{ig_s}{2}\lambda^\beta\gamma^\nu\right)c_4 v(4)\right],$$

with $q = p_1 + p_2 = p_3 + p_4$. Or,

$$\mathscr{M} = -\frac{g_s^2}{4q^2}\left[\bar{v}(2)\gamma^\mu u(1)\right]\left[\bar{u}(3)\gamma_\mu v(4)\right]\left(c_2^\dagger\lambda^\alpha c_1\right)\left(c_3^\dagger\lambda^\alpha c_4\right).$$

Comparing the QED result (Eq. 7.109), we obtain the color factor

$$f = \boxed{\frac{1}{4}\left(c_2^\dagger \lambda^\alpha c_1\right)\left(c_3^\dagger \lambda^\alpha c_4\right)}.$$

In the singlet configuration,

$$\begin{aligned} f &= \frac{1}{4}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\left[(1\ 0\ 0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix} + (0\ 1\ 0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix} + (0\ 0\ 1)\,\lambda^\alpha \begin{pmatrix}0\\0\\1\end{pmatrix}\right] \\ &\quad \times \left[(1\ 0\ 0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix} + (0\ 1\ 0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix} + (0\ 0\ 1)\,\lambda^\alpha \begin{pmatrix}0\\0\\1\end{pmatrix}\right] \\ &= \frac{1}{12}(\mathrm{Tr}\lambda^\alpha)(\mathrm{Tr}\lambda^\alpha) = \boxed{0}, \end{aligned}$$

since the lambda matrices are all traceless (Eq. 8.34)—a color singlet cannot couple to a color octet (the gluon).

Problem 8.18

Starting with Eq. 8.57:

$$\begin{aligned} f &= \frac{1}{4}\left[c_3^\dagger \lambda^\alpha c_1\right]\left[c_4^\dagger \lambda^\alpha c_2\right] \\ &= \frac{1}{4}\frac{1}{\sqrt{2}}\left\{\left[c_3^\dagger \lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[c_4^\dagger \lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right] + \left[c_3^\dagger \lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[c_4^\dagger \lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right]\right\} \\ &= \frac{1}{4}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left\{\left[(1\ 0\ 0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(0\ 1\ 0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right]\right. \\ &\qquad + \left[(0\ 1\ 0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(1\ 0\ 0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right] \\ &\qquad + \left[(1\ 0\ 0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[(0\ 1\ 0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right] \\ &\qquad \left. + \left[(0\ 1\ 0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[(1\ 0\ 0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right]\right\} \end{aligned}$$

$$\begin{aligned}
f &= \frac{1}{8}\left(\lambda^\alpha_{11}\lambda^\alpha_{22} + \lambda^\alpha_{21}\lambda^\alpha_{12} + \lambda^\alpha_{12}\lambda^\alpha_{21} + \lambda^\alpha_{22}\lambda^\alpha_{11}\right) = \frac{1}{4}\left(\lambda^\alpha_{11}\lambda^\alpha_{22} + \lambda^\alpha_{12}\lambda^\alpha_{21}\right) \\
&= \frac{1}{4}\left(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22} + \lambda^1_{12}\lambda^1_{21} + \lambda^2_{12}\lambda^2_{21}\right) \\
&= \frac{1}{4}\left[(1)(-1) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + (1)(1) + (-i)(i)\right] \\
&= \frac{1}{4}\left(-1 + \frac{1}{3} + 1 + 1\right) = \frac{1}{4}\left(\frac{4}{3}\right) = \boxed{\tfrac{1}{3}}.
\end{aligned}$$

Problem 8.19

$$\begin{aligned}
(\mathbf{a})\quad \lambda^\alpha_{11}\lambda^\alpha_{11} &= \lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11} = (1)(1) + (1/\sqrt{3})(1/\sqrt{3}) = 1 + \frac{1}{3} = \frac{4}{3} \\
&\overset{?}{=} 2(1)(1) - \tfrac{2}{3}(1)(1) = 2 - \frac{2}{3} = \frac{4}{3}. \quad \checkmark \\
(\mathbf{b})\quad \lambda^\alpha_{11}\lambda^\alpha_{22} &= \lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22} = (1)(-1) + (\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{3}}) = -1 + \frac{1}{3} = -\frac{2}{3} \\
&\overset{?}{=} 0 - \tfrac{2}{3}(1)(1) = -\frac{2}{3}. \quad \checkmark \\
(\mathbf{c})\quad \lambda^\alpha_{12}\lambda^\alpha_{21} &= \lambda^1_{12}\lambda^1_{21} + \lambda^2_{12}\lambda^2_{21} = (1)(1) + (-i)(i) = 1 + 1 = 2 \\
&\overset{?}{=} 2(1)(1) - 0 = 2. \quad \checkmark \\
(\mathbf{d})\quad \mathrm{Tr}\,(\lambda^\alpha\lambda^\alpha) &= \sum_{i,j}\lambda^\alpha_{ij}\lambda^\alpha_{ji} = \sum_{i,j}\left(2\delta_{ii}\delta_{jj} - \tfrac{2}{3}\delta_{ij}\delta_{ji}\right) \\
&= 2\sum_{i=1}^{3}\delta_{ii}\sum_{j=1}^{3}\delta_{jj} - \frac{2}{3}\sum_{i=1}^{3}\delta_{ii} = 2(3)(3) - \frac{2}{3}(3) = 18 - 2 = 16. \checkmark
\end{aligned}$$

Problem 8.20

Start with Eq. 8.69:

$$\begin{aligned}
\mathscr{M}_3 = i\bar{v}(2)c_2^\dagger &\left[-i\frac{g_s}{2}\lambda^\delta\gamma_\sigma\right]u(1)c_1\left[-i\frac{g^{\sigma\lambda}\delta^{\delta\gamma}}{q^2}\right]\left[-g_s f^{\alpha\beta\gamma}\right] \\
&\times\ [g_{\mu\nu}(-p_3 + p_4)_\lambda + g_{\nu\lambda}(-p_4 - q)_\mu + g_{\lambda\mu}(q + p_3)_\nu][\epsilon_3^\mu a_3^\alpha][\epsilon_4^\nu a_4^\beta] \\
= i\frac{g_s^2}{2q^2} &\left[\bar{v}(2)\gamma^\lambda u(1)\right]\left(f^{\alpha\beta\gamma}a_3^\alpha a_4^\beta\right)\left[c_2^\dagger\lambda^\gamma c_1\right] \\
&\times\ [(\epsilon_3\cdot\epsilon_4)(-p_3 + p_4)_\lambda - \epsilon_3\cdot(p_4 + q)\epsilon_{4\lambda} + \epsilon_4\cdot(p_3 + q)\epsilon_{3\lambda}]
\end{aligned}$$

Now,

$$q = p_3 + p_4, \quad q^2 = 2p_3 \cdot p_4, \quad \epsilon_3 \cdot p_3 = \epsilon_4 \cdot p_4 = 0,$$

so

$$\epsilon_3 \cdot (p_4 + q) = 2\epsilon_3 \cdot p_4, \quad \epsilon_4 \cdot (p_3 + q) = 2\epsilon_4 \cdot p_3,$$

and hence

$$\mathscr{M}_3 = i\frac{g_s^2}{4(p_3 \cdot p_4)} \{\bar{v}(2) [(\epsilon_3 \cdot \epsilon_4)(\not{p}_4 - \not{p}_3) - 2\,(\epsilon_3 \cdot p_4)\not{\epsilon}_4 + 2(\epsilon_4 \cdot p_3)\not{\epsilon}_3]\, u(1)\} \\ \times \left(f^{\alpha\beta\gamma} a_3^\alpha a_4^\beta\right) \left[c_2^\dagger \lambda^\gamma c_1\right]. \quad \checkmark$$

Problem 8.21

Replace ϵ_3 with p_3 in Eq. 8.67:

$$\mathcal{M}_1 \to -\frac{g_s^2}{8}\frac{1}{p_1 \cdot p_3} \{\bar{v}(2) [\not{\epsilon}_4(\not{p}_1 - \not{p}_3 + mc)\not{p}_3]\, u(1)\}\, a_3^\alpha a_4^\beta \left(c_2^\dagger \lambda^\beta \lambda^\alpha c_1\right).$$

Now $(\not{p}_1 - mc)u(1) = 0$, so

$$(\not{p}_1 - \not{p}_3 + mc)\not{p}_3 u(1) = (\not{p}_1 - \not{p}_3 + mc)(\not{p}_3 - \not{p}_1 + mc)u(1) \\ = -\left[(\not{p}_1 - \not{p}_3)^2 - (mc)^2\right] u(1) = \left[-(p_1 - p_3)^2 + (mc)^2\right] u(1)$$

(I used $\not{a}^2 = \not{a}\not{a} = a^2$.) But

$$-(p_1 - p_3)^2 + (mc)^2 = -p_1^2 - p_3^2 + 2(p_1 \cdot p_3) + (mc)^2 \\ = -\,(mc)^2 - 0 + 2(p_1 \cdot p_3) + (mc)^2 = 2(p_1 \cdot p_3)$$

So

$$\mathcal{M}_1 \to -\frac{g_s^2}{4}\,[\bar{v}(2)\not{\epsilon}_4 u(1)]\, a_3^\alpha a_4^\beta \left(c_2^\dagger \lambda^\beta \lambda^\alpha c_1\right).$$

Similarly, replacing ϵ_3 with p_3 in Eq. 8.68:

$$\mathcal{M}_2 \to -\frac{g_s^2}{8}\frac{1}{p_1 \cdot p_4} \{\bar{v}(2) [\not{p}_4(\not{p}_1 - \not{p}_4 + mc)\not{\epsilon}_4]\, u(1)\}\, a_3^\alpha a_4^\beta \left(c_2^\dagger \lambda^\alpha \lambda^\beta c_1\right).$$

But $\bar{v}(2)(\not{p}_2 + mc) = 0$, so

$$\bar{v}(2)\not{p}_3(\not{p}_1 - \not{p}_4 + mc) = \bar{v}(2)(\not{p}_3 - \not{p}_2 - mc)(\not{p}_1 - \not{p}_4 + mc) \\ = \bar{v}(2)(\not{p}_1 - \not{p}_4 - mc)(\not{p}_1 - \not{p}_4 + mc) = \bar{v}(2)\left[(\not{p}_1 - \not{p}_4)^2 - (mc)^2\right] \\ = \bar{v}(2)\left[(p_1 - p_4)^2 - (mc)^2)\right] = \bar{v}(2)\,[-2(p_1 \cdot p_4)]\,.$$

So

$$\mathcal{M}_2 \to \frac{g_s^2}{4}\left[\bar{v}(2)\not{\epsilon}_4 u(1)\right] a_3^\alpha a_4^\beta \left(c_2^\dagger \lambda^\alpha \lambda^\beta c_1\right).$$

Note that this *would* cancel the change in $\mathcal{M}_1$, except that the lambda matrices are in the opposite order:

$$\mathcal{M}_1 + \mathcal{M}_2 \to \frac{g_s^2}{4}\left[\bar{v}(2)\not{\epsilon}_4 u(1)\right] a_3^\alpha a_4^\beta \left(c_2^\dagger [\lambda^\alpha, \lambda^\beta] c_1\right).$$

But $[\lambda^\alpha, \lambda^\beta] = 2if^{\alpha\beta\gamma}\lambda^\gamma$ (Eq. 8.35), so

$$\mathcal{M}_1 + \mathcal{M}_2 \to i\frac{g_s^2}{2}\left[\bar{v}(2)\not{\epsilon}_4 u(1)\right] a_3^\alpha a_4^\beta f^{\alpha\beta\gamma} \left(c_2^\dagger \lambda^\gamma c_1\right).$$

Meanwhile, (Eq. 8.70)

$$\begin{aligned}
\mathcal{M}_3 \to i\frac{g_s^2}{4}\frac{1}{p_3 \cdot p_4} &\left\{\bar{v}(2)\left[(p_3 \cdot \epsilon_4)(\not{p}_4 - \not{p}_3) + 2(p_3 \cdot \epsilon_4)\not{p}_3 - 2(p_4 \cdot p_3)\not{\epsilon}_4\right] u(1)\right\} \\
&\times a_3^\alpha a_4^\beta f^{\alpha\beta\gamma}\left(c_2^\dagger \lambda^\gamma c_1\right) \\
= i\frac{g_s^2}{4}\frac{1}{p_3 \cdot p_4}&\bar{v}(2)\left[(p_3 \cdot \epsilon_4)(\not{p}_4 + \not{p}_3) - 2(p_4 \cdot p_3)\not{\epsilon}_4\right] u(1) a_3^\alpha a_4^\beta f^{\alpha\beta\gamma}\left(c_2^\dagger \lambda^\gamma c_1\right)
\end{aligned}$$

But

$$\bar{v}(2)(\not{p}_4 + \not{p}_3)u(1) = \bar{v}(2)(\not{p}_2 + \not{p}_1)u(1) = \bar{v}(2)(mc - mc)u(1) = 0,$$

so

$$\mathcal{M}_3 \to -i\frac{g_s^2}{2}\left[\bar{v}(2)\not{\epsilon}_4 u(1)\right] a_3^\alpha a_4^\beta f^{\alpha\beta\gamma}\left(c_2^\dagger \lambda^\gamma c_1\right),$$

which precisely cancels the change in $\mathcal{M}_1 + \mathcal{M}_2$. ✓

Problem 8.22

Most of the work is done in the statement of the problem. Filling in the details at the end:

$$\begin{aligned}
\mathrm{Tr}(N_1 N_2) &= \mathrm{Tr}\left\{\left[M_1 - \tfrac{1}{3}\mathrm{Tr}(M_1)\right]\left[M_2 - \tfrac{1}{3}\mathrm{Tr}(M_2)\right]\right\} \\
&= \mathrm{Tr}\left(M_1 M_2\right) - \tfrac{2}{3}\mathrm{Tr}\left(M_1\right)\mathrm{Tr}\left(M_2\right) + \tfrac{1}{9}\mathrm{Tr}\left(M_1\right)\mathrm{Tr}\left(M_2\right)\mathrm{Tr}\left(1\right).
\end{aligned}$$

But these are 3×3 matrices, so Tr(1) = 3:

$$\mathrm{Tr}(N_1 N_2) = \mathrm{Tr}\left(M_1 M_2\right) - \tfrac{1}{3}\mathrm{Tr}\left(M_1\right)\mathrm{Tr}\left(M_2\right).$$

Now

$$
\begin{aligned}
M_1M_2 &= \begin{pmatrix} (r\bar{r})_1 & (r\bar{b})_1 & (r\bar{g})_1 \\ (b\bar{r})_1 & (b\bar{b})_1 & (b\bar{g})_1 \\ (g\bar{r})_1 & (g\bar{b})_1 & (g\bar{g})_1 \end{pmatrix} \begin{pmatrix} (r\bar{r})_2 & (r\bar{b})_2 & (r\bar{g})_2 \\ (b\bar{r})_2 & (b\bar{b})_2 & (b\bar{g})_2 \\ (g\bar{r})_2 & (g\bar{b})_2 & (g\bar{g})_2 \end{pmatrix} \\
&= \begin{pmatrix} \left[(r\bar{r})_1(r\bar{r})_2 + (r\bar{b})_1(b\bar{r})_2 + (r\bar{g})_1(g\bar{r})_2\right] & \cdot & \cdot \\ \cdot & \left[(b\bar{r})_1(r\bar{b})_2 + (b\bar{b})_1(b\bar{b})_2 + (b\bar{g})_1(g\bar{b})_2\right] & \cdot \\ \cdot & \cdot & \left[(g\bar{r})_1(r\bar{g})_2 + (g\bar{b})_1(b\bar{g})_2 + (g\bar{g})_1(g\bar{g})_2\right] \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}(M_1M_2) &= (r\bar{r})_1(r\bar{r})_2 + (r\bar{b})_1(b\bar{r})_2 + (r\bar{g})_1(g\bar{r})_2 + (b\bar{r})_1(r\bar{b})_2 + (b\bar{b})_1(b\bar{b})_2 \\
&\quad + (b\bar{g})_1(g\bar{b})_2 + (g\bar{r})_1(r\bar{g})_2 + (g\bar{b})_1(b\bar{g})_2 + (g\bar{g})_1(g\bar{g})_2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}(M_1)\mathrm{Tr}(M_2) &= \left[(r\bar{r})_1 + (b\bar{b})_1 + (g\bar{g})_1\right]\left[(r\bar{r})_2 + (b\bar{b})_2 + (g\bar{g})_2\right] \\
&= (r\bar{r})_1(r\bar{r})_2 + (r\bar{r})_1(b\bar{b})_2 + (r\bar{r})_1(g\bar{g})_2 + (b\bar{b})_1(r\bar{r})_2 \\
+(b\bar{b})_1(b\bar{b})_2 &+ (b\bar{b})_1(g\bar{g})_2 + (g\bar{g})_1(r\bar{r})_2 + (g\bar{g})_1(b\bar{b})_2 + (g\bar{g})_1(g\bar{g})_2
\end{aligned}
$$

So

$$
\begin{aligned}
\mathrm{Tr}(N_1N_2) &= (r\bar{r})_1(r\bar{r})_2 + (r\bar{b})_1(b\bar{r})_2 + (r\bar{g})_1(g\bar{r})_2 + (b\bar{r})_1(r\bar{b})_2 + (b\bar{b})_1(b\bar{b})_2 \\
&\quad + (b\bar{g})_1(g\bar{b})_2 + (g\bar{r})_1(r\bar{g})_2 + (g\bar{b})_1(b\bar{g})_2 + (g\bar{g})_1(g\bar{g})_2 \\
&\quad - \tfrac{1}{3}\Big[(r\bar{r})_1(r\bar{r})_2 + (r\bar{r})_1(b\bar{b})_2 + (r\bar{r})_1(g\bar{g})_2 + (b\bar{b})_1(r\bar{r})_2 \\
&\quad + (b\bar{b})_1(b\bar{b})_2 + (b\bar{b})_1(g\bar{g})_2 + (g\bar{g})_1(r\bar{r})_2 + (g\bar{g})_1(b\bar{b})_2 + (g\bar{g})_1(g\bar{g})_2\Big] \\
&= \tfrac{2}{3}\Big[(r\bar{r})_1(r\bar{r})_2 + (b\bar{b})_1(b\bar{b})_2 + (g\bar{g})_1(g\bar{g})_2\Big] \\
&\quad + \Big[(r\bar{b})_1(b\bar{r})_2 + (r\bar{g})_1(g\bar{r})_2 + (b\bar{r})_1(r\bar{b})_2 + (b\bar{g})_1(g\bar{b})_2 + (g\bar{r})_1(r\bar{g})_2 \\
&\qquad + (g\bar{b})_1(b\bar{g})_2\Big] \\
&\quad - \tfrac{1}{3}\Big[(r\bar{r})_1(b\bar{b})_2 + (r\bar{r})_1(g\bar{g})_2 + (b\bar{b})_1(r\bar{r})_2 + (b\bar{b})_1(g\bar{g})_2 + (g\bar{g})_1(r\bar{r})_2 \\
&\qquad + (g\bar{g})_1(b\bar{b})_2\Big]
\end{aligned}
$$

Meanwhile (Eq. 8.29),

$$
\begin{aligned}
&\sum_{n=1}^{8} |n1\rangle_1|n\rangle_2 = \tfrac{1}{2}\left[(r\bar{b})_1 + (b\bar{r})_1\right]\left[(r\bar{b})_2 + (b\bar{r})_2\right] \\
&- \tfrac{1}{2}\left[(r\bar{b})_1 - (b\bar{r})_1\right]\left[(r\bar{b})_2 - (b\bar{r})_2\right] + \tfrac{1}{2}\left[(r\bar{r})_1 - (b\bar{b})_1\right]\left[(r\bar{r})_2 - (b\bar{b})_2\right] \\
&+ \tfrac{1}{2}\left[(r\bar{g})_1 + (g\bar{r})_1\right]\left[(r\bar{g})_2 + (g\bar{r})_2\right] - \tfrac{1}{2}\left[(r\bar{g})_1 - (g\bar{r})_1\right]\left[(r\bar{g})_2 - (g\bar{r})_2\right] \\
&+ \tfrac{1}{2}\left[(b\bar{g})_1 + (g\bar{b})_1\right]\left[(b\bar{g})_2 + (g\bar{b})_2\right] - \tfrac{1}{2}\left[(b\bar{g})_1 - (g\bar{b})_1\right]\left[(b\bar{g})_2 - (g\bar{b})_2\right] \\
&+ \tfrac{1}{6}\left[(r\bar{r})_1 + (b\bar{b})_1 - 2(g\bar{g})_1\right]\left[(r\bar{r})_2 + (b\bar{b})_2 - 2(g\bar{g})_2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{2}\Big\{(r\bar{b})_1(r\bar{b})_2 + (r\bar{b})_1(b\bar{r})_2 + (b\bar{r})_1(r\bar{b})_2 + (b\bar{r})_1(b\bar{r})_2 \\
&\quad - (r\bar{b})_1(r\bar{b})_2 + (r\bar{b})_1(b\bar{r})_2 + (b\bar{r})_1(r\bar{b})_2 - (b\bar{r})_1(b\bar{r})_2 \\
&\quad + (r\bar{r})_1(r\bar{r})_2 - (r\bar{r})_1(b\bar{b})_2 - (b\bar{b})_1(r\bar{r})_2 + (b\bar{b})_1(b\bar{b})_2 \\
&\quad + (r\bar{g})_1(r\bar{g})_2 + (r\bar{g})_1(g\bar{r})_2 + (g\bar{r})_1(r\bar{g})_2 + (g\bar{r})_1(g\bar{r})_2 \\
&\quad - (r\bar{g})_1(r\bar{g})_2 + (r\bar{g})_1(g\bar{r})_2 + (g\bar{r})_1(r\bar{g})_2 - (g\bar{r})_1(g\bar{r})_2 \\
&\quad + (b\bar{g})_1(b\bar{g})_2 + (b\bar{g})_1(g\bar{b})_2 + (g\bar{b})_1(b\bar{g})_2 + (g\bar{b})_1(g\bar{b})_2 \\
&\quad - (b\bar{g})_1(b\bar{g})_2 + (b\bar{g})_1(g\bar{b})_2 + (g\bar{b})_1(b\bar{g})_2 - (g\bar{b})_1(g\bar{b})_2 \\
&\quad + \tfrac{1}{3}\big[(r\bar{r})_1(r\bar{r})_2 + (r\bar{r})_1(b\bar{b})_2 - 2(r\bar{r})_1(g\bar{g})_2 + (b\bar{b})_1(r\bar{r})_2 + (b\bar{b})_1(b\bar{b})_2 \\
&\qquad - 2(b\bar{b})_1(g\bar{g})_2 - 2(g\bar{g})_1(r\bar{r})_2 - 2(g\bar{g})_1(b\bar{b})_2 + 4(g\bar{g})_1(g\bar{g})_2\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[(r\bar{b})_1(b\bar{r})_2 + (b\bar{r})_1(r\bar{b})_2 + (r\bar{g})_1(g\bar{r})_2 + (g\bar{r})_1(r\bar{g})_2 + (b\bar{g})_1(g\bar{b})_2 \\
&\qquad + (g\bar{b})_1(b\bar{g})_2\Big] + \tfrac{2}{3}\Big[(r\bar{r})_1(r\bar{r})_2 + (b\bar{b})_1(b\bar{b})_2 + (g\bar{g})_1(g\bar{g})_2\Big] \\
&\quad - \tfrac{1}{3}\Big[(r\bar{r})_1(b\bar{b})_2 + (r\bar{r})_1(g\bar{g})_2 + (b\bar{b})_1(r\bar{r})_2 + (b\bar{b})_1(g\bar{g})_2 + (g\bar{g})_1(r\bar{r})_2 \\
&\qquad + (g\bar{g})_1(b\bar{b})_2\Big] = \mathrm{Tr}(N_1N_2). \quad \checkmark
\end{aligned}
$$

Problem 8.23

From Eq. 8.90,

$$\Gamma(\eta_c \to 2g) = \frac{8\pi}{3c}\left(\frac{\hbar\alpha_s}{m}\right)^2 |\psi(0)|^2.$$

For the decay of positronium we had (Eq. 7.168 and 7.171)

$$\sigma = \frac{4\pi}{cv}\left(\frac{\hbar\alpha}{m}\right)^2 \quad \Rightarrow \quad \Gamma = \frac{4\pi}{c}\left(\frac{\hbar\alpha}{m}\right)^2 |\psi(0)|^2.$$

In the case of η_c the electron/positron are replaced by $c/\bar{c}$, so the charge $-e \to \frac{2}{3}e$, and hence $\alpha \to \frac{4}{9}\alpha$. Meanwhile, the amplitude picks up a factor of $1/\sqrt{3}$ (from Eq. 8.30) for each of the three $q\bar{q}$ pairs, and there are three of these, so the color factor in the amplitude is $3/\sqrt{3} = \sqrt{3}$, so Eq. 7.168 is multiplied by 3—and Eq. 7.171 inherits this factor:

$$\Gamma(\eta_c \to 2\gamma) = 3\left(\frac{4}{9}\right)^2 \frac{4\pi}{c}\left(\frac{\hbar\alpha}{m}\right)^2 |\psi(0)|^2 = \left(\frac{8}{9}\right)\frac{4\pi}{c}\left(\frac{\hbar\alpha}{m}\right)^2 |\psi(0)|^2.$$

Thus

$$\frac{\Gamma(\eta_c \to 2g)}{\Gamma(\eta_c \to 2\gamma)} = \boxed{\left(\frac{9}{8}\right)\left(\frac{\alpha_s}{\alpha}\right)^2}.$$

Problem 8.24

(a) $\alpha \to \infty$ when the denominator $\to 0$:

$$\left(\frac{\alpha}{3\pi}\right) \ln\left(\frac{|q|^2}{(mc)^2}\right) = 1 \quad \text{or} \qquad \frac{|q|^2}{(mc)^2} = e^{3\pi/\alpha}$$

$$\begin{aligned} E &= |q|c = mc^2 e^{3\pi/2\alpha} = (0.511)e^{3\pi(137)/2} = (0.511)e^{646} \\ &= (0.511) \times 10^{646 \log e} = \boxed{0.5 \times 10^{281} \text{MeV}}. \end{aligned}$$

(b)

$$\frac{\alpha(|q^2|) - \alpha(0)}{\alpha(0)} = \epsilon \quad \Rightarrow \quad \alpha(|q|^2) = (1+\epsilon)\alpha(0)$$

$$\frac{\alpha(0)}{1 - [\alpha(0)/3\pi] \ln[|q|^2/(mc)^2]} = (1+\epsilon)\alpha(0)$$

$$\frac{1}{1+\epsilon} \approx 1 - \epsilon = 1 - [\alpha/3\pi] \ln[|q|^2/(mc)^2]$$

$$\frac{\alpha}{3\pi} \ln\left[\frac{|q|^2}{(mc)^2}\right] = \epsilon \quad \Rightarrow \quad \ln\left[\frac{|q|^2}{(mc)^2}\right] = \frac{3\pi\epsilon}{\alpha}$$

$$\frac{|q|^2}{(mc)^2} = e^{3\pi\epsilon/\alpha}, \quad E = |q|c = mc^2 e^{3\pi\epsilon/2\alpha}.$$

For $\epsilon = 0.01$,

$$E = (0.511)e^{0.03\pi(137)/2} = (0.511)e^{6.46} = \boxed{326 \text{ MeV}}.$$

Yes, this is an accessible energy.

Problem 8.25

It is simplest to work with the *inverse* of Eq. 8.93. We assume this is valid for μ_a:

$$\frac{1}{\alpha_s(|q|^2)} = \frac{1}{\alpha_s(\mu_a^2)} + \frac{(11n - 2f)}{12\pi} \ln\left(\frac{|q|^2}{\mu_a^2}\right).$$

It follows in particular that

$$\frac{1}{\alpha_s(\mu_b^2)} = \frac{1}{\alpha_s(\mu_a^2)} + \frac{(11n - 2f)}{12\pi} \ln\left(\frac{\mu_b^2}{\mu_a^2}\right).$$

Solve this for $1/\alpha_s(\mu_a^2)$, and substitute back into the first equation:

$$\frac{1}{\alpha_s(|q|^2)} = \frac{1}{\alpha_s(\mu_b^2)} - \frac{(11n-2f)}{12\pi}\ln\left(\frac{\mu_b^2}{\mu_a^2}\right) + \frac{(11n-2f)}{12\pi}\ln\left(\frac{|q|^2}{\mu_a^2}\right)$$

$$= \frac{1}{\alpha_s(\mu_b^2)} + \frac{(11n-2f)}{12\pi}\ln\left(\frac{|q|^2}{\mu_a^2}\frac{\mu_a^2}{\mu_b^2}\right) = \frac{1}{\alpha_s(\mu_b^2)} + \frac{(11n-2f)}{12\pi}\ln\left(\frac{|q|^2}{\mu_b^2}\right).$$

So (the inverse of) Eq. 8.93 also holds for μ_b. ✓

Problem 8.26

Starting with Eq. 8.93, use $\ln(x/y) = \ln x - \ln y$ in the denominator:

$$\alpha_s(|q|^2) = \frac{\alpha_s(\mu^2)}{1 + [\alpha_s(\mu^2)/12\pi]\,(11n-2f)\ln|q|^2 - [\alpha_s(\mu^2)/12\pi]\,(11n-2f)\ln\mu^2}.$$

Substitute $\ln\Lambda^2 + 12\pi/[(11n-2f)\alpha_s(\mu^2)]$ for $\ln\mu^2$ (Eq. 8.94) in the last term:

$$\alpha_s(|a|^2) = \frac{\alpha_s(\mu^2)}{1 + [\alpha_s(\mu^2)/12\pi]\,(11n-2f)\ln(|q|^2/\Lambda^2) - 1}$$

$$= \frac{12\pi}{(11n-2f)\ln(|q|^2/\Lambda^2)}. \quad ✓$$

Problem 8.27

Equation 8.95, with $n = 3$ and $f = 6$ (the Standard Model values) gives

$$\alpha_s = \frac{12\pi}{21\ln(|q|c/\Lambda c)^2} = \frac{2\pi}{7\ln(|q|c/\Lambda c)}.$$

(This assumes $|q|c \gg \Lambda c$, which is true for all the values used here.)

$\Lambda c = 0.3$

$$10\text{ GeV} \qquad \alpha_s = \frac{2\pi}{7\ln(10/0.3)} = \boxed{0.256}$$

$$100\text{ GeV} \qquad \alpha_s = \frac{2\pi}{7\ln(100/0.3)} = \boxed{0.155}$$

$\Lambda c = 1$

$$10\text{ GeV} \qquad \alpha_s = \frac{2\pi}{7\ln(10/1)} = \boxed{0.390}$$

$$100\text{ GeV} \qquad \alpha_s = \frac{2\pi}{7\ln(100/1)} = \boxed{0.195}$$

$\Lambda c = 0.1$

$$10\,\text{GeV} \qquad \alpha_s = \frac{2\pi}{7\ln(10/0.1)} = \boxed{0.195}$$

$$100\,\text{GeV} \qquad \alpha_s = \frac{2\pi}{7\ln(100/0.1)} = \boxed{0.130}$$

Problem 8.28

9
Weak Interactions

Problem 9.1

$$p^\mu = (p^0, 0, 0, |\mathbf{p}|); \qquad (p^0)^2 - \mathbf{p}^2 = M^2c^2.$$

$$\epsilon^{(1)}_\mu = (0,1,0,0), \quad \epsilon^{(2)}_\mu = (0,0,1,0), \quad \epsilon^{(3)}_\mu = (a,0,0,b).$$

$$\epsilon^{(3)}_\mu p^\mu = ap^0 + b|\mathbf{p}| = 0 \;\Rightarrow b = -\left(\frac{p^0}{|\mathbf{p}|}\right) a;$$

$$\epsilon^{(3)}_\mu \epsilon^{(3)\mu} = a^2 - b^2 = \frac{a^2}{\mathbf{p}^2}\left(\mathbf{p}^2 - p^{0\,2}\right) = \frac{a^2}{\mathbf{p}^2}\left(-M^2c^2\right) = -1$$

$$a^2 = \frac{\mathbf{p}^2}{(Mc)^2} \Rightarrow a = \frac{|\mathbf{p}|}{Mc},\; b = -\frac{p^0}{Mc},\; \epsilon^{(3)}_\mu = \frac{1}{Mc}(|\mathbf{p}|, 0, 0, -p^0).$$

$$\sum_s \epsilon^{(s)}_0 \epsilon^{(s)*}_0 = 0 + 0 + a^2 = \frac{|\mathbf{p}|^2}{(Mc)^2}, \qquad -g_{00} + \frac{p_0 p_0}{(Mc)^2)} = \frac{|\mathbf{p}|^2}{(Mc)^2};$$

$$\sum_s \epsilon^{(s)}_1 \epsilon^{(s)*}_1 = 1 + 0 + 0 = 1, \qquad -g_{11} + \frac{p_1 p_1}{(Mc)^2)} = 1;$$

$$\sum_s \epsilon^{(s)}_2 \epsilon^{(s)*}_2 = 0 + 1 + 0 = 1, \qquad -g_{22} + \frac{p_2 p_2}{(Mc)^2)} = 1;$$

$$\sum_s \epsilon^{(s)}_3 \epsilon^{(s)*}_3 = 0 + 0 + b^2 = \frac{(p^0)^2}{(Mc)^2}, \qquad -g_{33} + \frac{p_3 p_3}{(Mc)^2)} = \frac{(p^0)^2}{(Mc)^2};$$

$$\sum_s \epsilon^{(s)}_0 \epsilon^{(s)*}_3 = \sum_s \epsilon^{(s)}_3 \epsilon^{(s)*}_0 = ab = \frac{-p^0|\mathbf{p}|}{(Mc)^2}, \quad -g_{03} + \frac{p_0 p_3}{(Mc)^2)} = \frac{-p^0|\mathbf{p}|}{(Mc)^2}$$

(all other combinations are zero). This confirms Eq. 9.158.

Problem 9.2

Let $\epsilon \equiv c_A/c_V$, and drop the two terms involving an odd number of gammas:

$$T \equiv \mathrm{Tr}\left[\gamma^\mu(c_V - c_A\gamma^5)(\not{p}_1 + m_1c)\gamma^\nu(c_V - c_A\gamma^5)(\not{p}_2 + m_2c)\right] =$$

$$c_V^2\left\{\underbrace{\mathrm{Tr}\left[\gamma^\mu(1-\epsilon\gamma^5)\not{p}_1\gamma^\nu(1-\epsilon\gamma^5)\not{p}_2)\right]}_{T_1} + m_1m_2c^2\underbrace{\mathrm{Tr}\left[\gamma^\mu(1-\epsilon\gamma^5)\gamma^\nu(1-\epsilon\gamma^5)\right]}_{T_2}\right\}$$

$$\begin{aligned}
T_2 &= \underbrace{\mathrm{Tr}(\gamma^\mu\gamma^\nu)}_{4g^{\mu\nu}} - \epsilon\underbrace{\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5)}_{0} - \epsilon\underbrace{\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^\nu)}_{0} + \epsilon^2\underbrace{\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)}_{-\mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu\gamma^\nu) = -4g^{\mu\nu}} \\
&= 4g^{\mu\nu}(1-\epsilon^2). \\
T_1 &= \mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) - \epsilon\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu\gamma^5\not{p}_2) - \epsilon\mathrm{Tr}(\gamma^\mu\gamma^5\not{p}_1\gamma^\nu\not{p}_2) \\
&\quad + \epsilon^2\underbrace{\mathrm{Tr}(\gamma^\mu\gamma^5\not{p}_1\gamma^\nu\gamma^5\not{p}_2)}_{\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2)}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) &= p_{1\lambda}p_{2\sigma}\mathrm{Tr}(\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma) = 4p_{1\lambda}p_{2\sigma}(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\lambda\nu}) \\
&= 4(p_1^\mu p_2^\nu - g^{\mu\nu}p_1\cdot p_2 + p_1^\nu p_2^\mu). \\
\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu\gamma^5\not{p}_2) &= \mathrm{Tr}(\gamma^\mu\gamma^5\not{p}_1\gamma^\nu\not{p}_2) = -p_{1\lambda}p_{2\sigma}\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma) \\
&= -4i\epsilon^{\mu\lambda\nu\sigma}p_{1\lambda}p_{2\sigma}.
\end{aligned}$$

So

$$T_1 = 4(1+\epsilon^2)\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\right] - 8i\epsilon(p_{1\lambda}p_{2\sigma})\epsilon^{\mu\nu\lambda\sigma},$$

and

$$\begin{aligned}
T &= 4c_V^2\Big\{(1+\epsilon^2)\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\right] - 2i\epsilon(p_{1\lambda}p_{2\sigma})\epsilon^{\mu\nu\lambda\sigma} \\
&\quad + m_1m_2c^2g^{\mu\nu}(1-\epsilon^2)\Big\} \\
&= 4\left(c_V^2 + c_A^2\right)\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\right] + 4\left(c_V^2 - c_A^2\right)m_1m_2c^2g^{\mu\nu} \\
&\quad - 8ic_Vc_A\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{2\sigma}.
\end{aligned}$$

Problem 9.3

(a) In place of Eq. 9.6, we have

$$\mathcal{M} = \frac{g_w^2}{8(M_Wc)^2}\left[\bar{u}(3)\gamma^\mu(1+\epsilon\gamma^5)u(1)\right]\left[\bar{u}(4)\gamma_\mu(1+\epsilon\gamma^5)u(2)\right].$$

Casimir's trick runs as before, yielding (in place of Eq. 9.7):

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\left[\frac{g_w^2}{8(M_W c)^2}\right]^2 \underbrace{\text{Tr}\left[\gamma^\mu(1+\epsilon\gamma^5)(\not{p}_1 + m_e c)\gamma^\nu(1+\epsilon\gamma^5)\not{p}_3\right]}_{A^{\mu\nu}}$$

$$\times \underbrace{\text{Tr}\left[\gamma_\mu(1+\epsilon\gamma^5)\not{p}_2\gamma_\nu(1+\epsilon\gamma^5)(\not{p}_4 + m_\mu c)\right]}_{B_{\mu\nu}}$$

Using the result of Problem 9.2,

$$A^{\mu\nu} = 4(1+\epsilon^2)\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1\cdot p_3)\right] + 8i\epsilon\epsilon^{\mu\nu\lambda\sigma}p_{1_\lambda}p_{3_\sigma},$$
$$B_{\mu\nu} = 4(1+\epsilon^2)\left[p_{2_\mu}p_{4_\nu} + p_{2_\nu}p_{4_\mu} - g_{\mu\nu}(p_2\cdot p_4)\right] + 8i\epsilon\epsilon_{\mu\nu\phi\tau}p_2^\phi p_4^\tau.$$

Note that in each case the first term is *symmetric*, and the last *antisymmetric*, in $\mu \leftrightarrow \nu$. So

$$\begin{aligned}
A^{\mu\nu}B_{\mu\nu} &= -(8\epsilon)^2(\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\phi\tau})p_{1_\lambda}p_{3_\sigma}p_2^\phi p_4^\tau + \left[4(1+\epsilon^2)\right]^2 \\
&\times\left\{\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1\cdot p_3)\right]\left[p_{2_\mu}p_{4_\nu} + p_{2_\nu}p_{4_\mu} - g_{\mu\nu}(p_2\cdot p_4)\right]\right\} \\
&= -(8\epsilon)^2(-2)(\delta^\lambda_\phi\delta^\sigma_\tau - \delta^\lambda_\tau\delta^\sigma_\phi)p_{1_\lambda}p_{3_\sigma}p_2^\phi p_4^\tau + \left[4(1+\epsilon^2)\right]^2 \\
&\times\Big\{(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4) \\
&\quad + (p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) \\
&\quad - (p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) + 4(p_1\cdot p_3)(p_2\cdot p_4)\Big\} \\
&= \left[4(1+\epsilon^2)\right]^2[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)] \\
&\quad + 2(8\epsilon)^2[(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3)] \\
&= 32\Big\{(1+\epsilon^2)^2[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)] \\
&\quad + 4\epsilon^2[(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3)]\Big\} \\
&= 32\Big\{(p_1\cdot p_2)(p_3\cdot p_4)(1+2\epsilon^2+\epsilon^4+4\epsilon^2) \\
&\quad + (p_1\cdot p_4)(p_2\cdot p_3)(1+2\epsilon^2+\epsilon^4-4\epsilon^2)\Big\} \\
&= 32\left[(1+6\epsilon^2+\epsilon^4)(p_1\cdot p_2)(p_3\cdot p_4) + (1-\epsilon^2)^2(p_1\cdot p_4)(p_2\cdot p_3)\right].
\end{aligned}$$

$$\langle|\mathcal{M}|^2\rangle = \boxed{\frac{1}{4}\left(\frac{g_w}{M_W c}\right)^4\left[(1+6\epsilon^2+\epsilon^4)(p_1\cdot p_2)(p_3\cdot p_4) + (1-\epsilon^2)^2(p_1\cdot p_4)(p_2\cdot p_3)\right]}$$

When $\epsilon = -1$, the second term drops out, and $(1 + 6\epsilon^2 + \epsilon^4) = 8$, so

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_w}{M_W c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4),$$

in agreement with Eq. 9.11.

(b) In the *CM* frame, with $m_e = m_\mu = 0$:

$$p_1 = \left(\frac{E}{c}, \mathbf{p}_1\right);\ p_2 = \left(\frac{E}{c}, -\mathbf{p}_1\right);\ p_3 = \left(\frac{E}{c}, -\mathbf{p}_4\right);\ p_4 = \left(\frac{E}{c}, \mathbf{p}_4\right).$$

So

$$(p_1 \cdot p_2) = \frac{E^2}{c^2} + \mathbf{p}_1^2 = \frac{E^2}{c^2} + \frac{E^2}{c^2} = 2\frac{E^2}{c^2} = (p_3 \cdot p_4),$$
$$(p_1 \cdot p_4) = \frac{E^2}{c^2} - \mathbf{p}_1 \cdot \mathbf{p}_4 = \frac{E^2}{c^2} - \mathbf{p}^2 \cos\theta = \frac{E^2}{c^2}(1 - \cos\theta) = (p_2 \cdot p_3),$$

where θ is the angle between $\mathbf{p}_1$ (the incoming electron) and $\mathbf{p}_4$ (the outgoing muon).

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\left(\frac{g_w}{M_W c}\right)^4 \left[(1 + 6\epsilon^2 + \epsilon^4)\left(2\frac{E^2}{c^2}\right)^2\right.\\ &\qquad \left. + (1 - \epsilon^2)^2 \left(\frac{E^2}{c^2}\right)^2 (1 - \cos\theta)^2\right]\\ &= \left(\frac{g_w E}{M_W c^2}\right)^4 \left[(1 + 6\epsilon^2 + \epsilon^4) + \frac{1}{4}(1 - \epsilon^2)^2(1 - \cos\theta)^2\right].\end{aligned}$$

From Eq. 6.47,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\langle|\mathcal{M}|^2\rangle}{(2E)^2}\\ &= \left[\left(\frac{g_w E}{M_W c^2}\right)^2 \frac{\hbar c}{8\pi}\frac{1}{2E}\right]^2 \left[(1 + 6\epsilon^2 + \epsilon^4) + \frac{1}{4}(1 - \epsilon^2)^2 \left(2\sin^2\frac{\theta}{2}\right)^2\right]\\ &= \boxed{\left(\frac{\hbar g_w^2 E}{16\pi M_W^2 c^3}\right)^2 \left[(1 + 6\epsilon^2 + \epsilon^4) + (1 - \epsilon^2)^2 \sin^4\frac{\theta}{2}\right]}.\end{aligned}$$

The *total* cross section is $\sigma = \int (d\sigma/d\Omega) d\Omega$. The first term integrates to 4π (no θ dependence). The second term gives

$$\begin{aligned}\int \sin^4\frac{\theta}{2} \sin\theta\, d\theta\, d\phi &= 2\pi\frac{1}{4}\int_0^\pi (1 - \cos\theta)^2 \sin\theta\, d\theta\\ &= \frac{\pi}{2}\cdot\frac{1}{3}(1 - \cos\theta)^3\Big|_0^\pi = \frac{\pi}{6}(2)^3 = \frac{4}{3}\pi.\end{aligned}$$

So

$$\sigma = \left(\frac{\hbar g_w^2 E}{16\pi M_W^2 c^3}\right)^2 4\pi \underbrace{\left[(1+6\epsilon^2+\epsilon^4)+(1-\epsilon^2)^2\frac{1}{3}\right]}_{\#}.$$

$$\# = \frac{1}{3}(3+18\epsilon^2+3\epsilon^4+1-2\epsilon^2+\epsilon^4) = \frac{1}{3}(4+16\epsilon^2+4\epsilon^4) = \frac{4}{3}(1+4\epsilon^2+\epsilon^4).$$

Evidently

$$\sigma = \boxed{\frac{1}{3\pi}\left(\frac{\hbar g_w^2 E}{4M_W^2 c^3}\right)^2 (1+4\epsilon^2+\epsilon^4)}.$$

(c) The *sign* of ϵ *cannot* be determined, unless you actually study the incoming/outgoing *spins*, because the spin-averaged amplitude only involves ϵ^2. You could determine ϵ^2 from the total cross section, if you had very good values for g_w and M_W. But it would be better to study the differential cross section *as a function of angle*. In *practice* you would find *no* angular dependence, and this would indicate that $\epsilon^2 = 1$.

Problem 9.4

Equation 9.28 says that $(|\mathbf{p}_2| - |\mathbf{p}_4|) \leq u_-$ and $(|\mathbf{p}_4| - |\mathbf{p}_2|) \leq u_-$ (since u_- is the greater of the two). So the first inequality in Eq. 9.29 holds if (and only if) the following *two* inequalities are true:

$$|\mathbf{p}_2| - \cancel{|\mathbf{p}_4|} < m_\mu - |\mathbf{p}_2| - \cancel{|\mathbf{p}_4|}, \quad \text{or} \quad |\mathbf{p}_2| < \tfrac{1}{2}m_\mu c,$$
$$|\mathbf{p}_4| - \cancel{|\mathbf{p}_2|} < m_\mu - \cancel{|\mathbf{p}_2|} - |\mathbf{p}_4|, \quad \text{or} \quad |\mathbf{p}_4| < \tfrac{1}{2}m_\mu c.$$

The second inequality in Eq. 9.29 holds $\Leftrightarrow$

$$m_\mu c - |\mathbf{p}_2| - |\mathbf{p}_4| < |\mathbf{p}_2| + |\mathbf{p}_4|, \quad \text{or} \quad |\mathbf{p}_2| + |\mathbf{p}_4| > \tfrac{1}{2}m_\mu c.$$

Problem 9.5

$\tau \to \mu + \bar{\nu}_\mu + \nu_\tau$ is just like $\mu \to e + \bar{\nu}_e + \nu_\mu$, so the rate is given by Eq. 9.34. (Note that $m_\mu/m_\tau = 0.059$, so it is reasonable to assume $m_\mu \ll m_\tau$.) The rate goes as the fifth power of the decaying particle's mass. But the τ can also go to $e + \bar{\nu}_e + \nu_\tau$, with the same rate. So

$$\frac{\tau(\tau \to \text{leptons})}{\tau(\mu \to \text{leptons})} = \frac{m_\mu^5}{2m_\tau^5}, \quad \text{or} \quad \boxed{\tau_\tau = \frac{1}{2}\left(\frac{m_\mu}{m_\tau}\right)^5 \tau_\mu}.$$

Numerically:

$$\tau_\tau = \frac{1}{2}\left(\frac{105.66}{1777}\right)^5 (2.197 \times 10^{-6}\ \text{s}) = \boxed{8.16 \times 10^{-13}\ \text{s}}.$$

The *actual* lifetime is 2.91×10^{-13} s. The discrepancy is due to the fact that the τ has many *hadronic* decay modes, in addition to the μ and e routes. In fact, the experimental branching ratios are 17.4% (μ) and 17.8% (e), for a total leptonic branching ratio of 35.2%, so $\tau = (0.352)\tau(\text{leptons}) = (0.352)(8.16 \times 10^{-13}\ \text{s}) = 2.87 \times 10^{-13}$ s, which is quite close to the experimental result.

Problem 9.6

For vector coupling Eq. 9.15 becomes

$$\mathcal{M} = \frac{g_w^2}{8(M_W c)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu v(2)\right],$$

which is a special case of Problem 9.3(a), with $\epsilon = 0$ (also $v(2)$ in place of $u(2)$, but since this is a neutrino, the completeness relation gives $\not{p}_2 \pm m_2 c = \not{p}_2$ in either case):

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\left(\frac{g_w}{M_W c}\right)^4 \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)\right]$$

(in place of Eq. 9.16).

In the muon rest frame (Eqs. 9.17 and 9.19)

$$p_1 \cdot p_2 = m_\mu E_2, \quad p_1 \cdot p_4 = m_\mu E_4, \quad p_3 \cdot p_4 = \frac{(m_\mu^2 - m_e^2)c^2}{2} - m_\mu E_2.$$

Also,

$$\begin{aligned}(p_2 + p_3)^2 &= p_2^2 + p_3^2 + 2p_2 \cdot p_3 = 2p_2 \cdot p_3 = (p_1 - p_4)^2 \\ &= p_1^2 + p_4^2 - 2p_1 \cdot p_4 = m_\mu^2 c^2 + m_e^2 c^2 - 2p_1 \cdot p_4.\end{aligned}$$

so

$$p_2 \cdot p_3 = \frac{(m_\mu^2 + m_e^2)c^2}{2} - m_\mu E_4.$$

Let $m_e = 0$ (as in the book); then

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\left(\frac{g_w}{M_W c}\right)^4 \left[m_\mu E_2 m_\mu\left(\tfrac{1}{2}m_\mu c^2 - E_2\right) + m_\mu E_4 m_\mu\left(\tfrac{1}{2}m_\mu c^2 - E_4\right)\right] \\ &= \frac{1}{8}\left(\frac{g_w^2 m_\mu}{M_W^2 c}\right)^2 \left[|\mathbf{p}_2|(m_\mu c - 2|\mathbf{p}_2|) + |\mathbf{p}_4|(m_\mu c - 2|\mathbf{p}_4|)\right].\end{aligned}$$

This is identical to Eq. 9.20, except for the overall factor of 1/8 and the extra term with $|\mathbf{p}_4|$ in place of $|\mathbf{p}_2|$. The $|\mathbf{p}_2|$ term yields (quoting Eq. 9.32)

$$d\Gamma_2 = \frac{1}{8}\left(\frac{g_w}{4\pi M_W}\right)^4 \frac{m_\mu}{\hbar c^2}\left(\frac{m_\mu c}{2} - \frac{2}{3}|\mathbf{p}_4|\right) d^3\mathbf{p}_4,$$

while the $|\mathbf{p}_4|$ term is (from the first line in Eq. 9.32)

$$\begin{aligned} d\Gamma_4 &= \frac{1}{8}\left(\frac{g_w}{4\pi M_W}\right)^4 \frac{m_\mu}{\hbar c^2}\frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2}|\mathbf{p}_4|(m_\mu c - 2|\mathbf{p}_4|)\underbrace{\int_{\frac{1}{2}m_\mu c - |\mathbf{p}_4|}^{\frac{1}{2}m_\mu c} d|\mathbf{p}_2|}_{|\mathbf{p}_4|} \\ &= \frac{1}{8}\left(\frac{g_w}{4\pi M_W}\right)^4 \frac{m_\mu}{\hbar c^2}\left(m_\mu c - 2|\mathbf{p}_4|\right) d^3\mathbf{p}_4. \end{aligned}$$

Combining the two,

$$d\Gamma = \frac{1}{8}\left(\frac{g_w}{4\pi M_W}\right)^4 \frac{m_\mu}{\hbar c^2}\left(\tfrac{3}{2}m_\mu c - \tfrac{8}{3}|\mathbf{p}_4|\right) 4\pi|\mathbf{p}_4|^2\, d|\mathbf{p}_4|,$$

or, in terms of the electron energy $E = |\mathbf{p}_4|c$:

$$\frac{d\Gamma}{dE} = \frac{3}{8}\left(\frac{g_w}{M_W c}\right)^4 \frac{m_\mu^2 E^2}{2\hbar(4\pi)^3}\left(1 - \frac{16E}{9m_\mu c^2}\right).$$

The dependence on E is *not* the same as before (Eq. 9.33)—16/9 in place of 4/3 in the last term—so the electron spectrum (Figure 9.1) would be $\boxed{\text{different}}$ for pure vector coupling.

Problem 9.7

Use Eq. 9.33, with $\beta \equiv \frac{1}{2}m_\mu c^2$:

$$\begin{aligned} \langle E\rangle &= \frac{\int E\left(\frac{d\Gamma}{dE}\right) dE}{\int \left(\frac{d\Gamma}{dE}\right) dE} = \frac{\int_0^\beta E^3\left(1 - \frac{2E}{3\beta}\right) dE}{\int_0^\beta E^2\left(1 - \frac{2E}{3\beta}\right) dE} = \frac{\left(\frac{1}{4}E^4 - \frac{1}{5}\frac{2}{3\beta}E^5\right)\Big|_0^\beta}{\left(\frac{1}{3}E^3 - \frac{1}{4}\frac{2}{3\beta}E^4\right)\Big|_0^\beta} \\ &= \frac{\beta^4\left(\frac{1}{4} - \frac{2}{15}\right)}{\beta^3\left(\frac{1}{3} - \frac{1}{6}\right)} = \beta\left(\frac{7/60}{1/6}\right) = \boxed{\frac{7}{20}m_\mu c^2}. \end{aligned}$$

Problem 9.8

$$\mathcal{M} = \frac{g_w^2}{8(M_W c)^2}\left[\bar{u}(3)\gamma^\mu(1 + \epsilon\gamma^5)u(1)\right]\left[\bar{u}(4)\gamma_\mu(1 - \gamma^5)v(2)\right].$$

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\left[\frac{g_w^2}{8(M_W c)^2}\right]^2 \underbrace{\text{Tr}\left[\gamma^\mu(1+\epsilon\gamma^5)(\not p_1 + m_1 c)\gamma^\nu(1+\epsilon\gamma^5)(\not p_3 + m_3 c)\right]}_{A^{\mu\nu}}$$
$$\times \underbrace{\text{Tr}\left[\gamma_\mu(1-\gamma^5)(\not p_2 - m_2 c)\gamma_\nu(1-\gamma^5)(\not p_4 + m_4 c)\right]}_{B_{\mu\nu}}$$

Using the result of Problem 9.2, with $c_V \to 1$, $\epsilon \to -\epsilon$, $2 \to 3$, $m_1 = m_n$ and $m_3 = m_p$:

$$A^{\mu\nu} = 4(1+\epsilon^2)\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1\cdot p_3)\right] + 8i\epsilon\epsilon^{\mu\nu\lambda\sigma}p_{1_\lambda}p_{3_\sigma}$$
$$+ 4m_p m_n c^2 g^{\mu\nu}(1-\epsilon^2),$$

and similarly, with μ and ν lowered, $\epsilon \to 1$, $1 \to 2$, $2 \to 4$, and $m_2 = m_4 = 0$:

$$B_{\mu\nu} = 8\left[p_{2_\mu}p_{4_\nu} + p_{2_\nu}p_{4_\mu} - g_{\mu\nu}(p_2\cdot p_4)\right] - 8i\epsilon_{\mu\nu\phi\tau}p_2^\phi p_4^\tau.$$

Note that in each case the first term is *symmetric*, the second *antisymmetric*, and (in $A^{\mu\nu}$) the third is *symmetric*, in $\mu \leftrightarrow \nu$. So $A^{\mu\nu}B_{\mu\nu}$ is

$$32\Big\{(1+\epsilon^2)\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1\cdot p_3)\right]\left[p_{2_\mu}p_{4_\nu} + p_{2_\nu}p_{4_\mu} - g_{\mu\nu}(p_2\cdot p_4)\right]$$
$$+ 2\epsilon(p_{1_\lambda}p_{3_\sigma}p_2^\phi p_4^\tau)\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\phi\tau}$$
$$+ (1-\epsilon^2)m_p m_n c^2 g^{\mu\nu}\left[p_{2_\mu}p_{4_\nu} + p_{2_\nu}p_{4_\mu} - g_{\mu\nu}(p_2\cdot p_4)\right]\Big\},$$

or (using Problem 7.35(b) for the middle term):

$$A^{\mu\nu}B_{\mu\nu} = 32\Big\{2(1+\epsilon^2)\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]$$
$$- 4\epsilon\left[(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3)\right]$$
$$- 2(1-\epsilon^2)m_p m_n c^2(p_2\cdot p_4)\Big\}$$
$$= 64\Big\{(p_1\cdot p_2)(p_3\cdot p_4)(1-\epsilon)^2 + (p_1\cdot p_4)(p_2\cdot p_3)(1+\epsilon)^2$$
$$- (1-\epsilon^2)m_p m_n c^2(p_2\cdot p_4)\Big\}$$

So

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4\Big\{(p_1\cdot p_2)(p_3\cdot p_4)(1-\epsilon)^2$$
$$+ (p_1\cdot p_4)(p_2\cdot p_3)(1+\epsilon)^2 - (1-\epsilon^2)m_p m_n c^2(p_2\cdot p_4)\Big\}.$$

When $\epsilon = -1$, this reduces to

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4\Big\{4(p_1\cdot p_2)(p_3\cdot p_4)\Big\},$$

in agreement with Eq. 9.41.

Problem 9.9

(a)

$$\begin{aligned}
u_-^2 &= \cancel{|\mathbf{p}_2|^2} + \cancel{|\mathbf{p}_4|^2} - 2|\mathbf{p}_2||\mathbf{p}_4| + m_p^2c^2 < \left(m_nc - |\mathbf{p}_2| - \sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}\right)^2 \\
&= m_n^2c^2 + \cancel{|\mathbf{p}_2|^2} + \cancel{|\mathbf{p}_4|^2} + m_e^2c^2 - 2m_nc|\mathbf{p}_2| - 2m_nc\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2} \\
&\quad + 2|\mathbf{p}_2|\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2};
\end{aligned}$$

$$2|\mathbf{p}_2|\left(m_nc - |\mathbf{p}_4| - \sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}\right) < (m_n^2 - m_p^2 + m_e^2)c^2 - 2m_nc\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2};$$

$$|\mathbf{p}_2| < \frac{\frac{1}{2}(m_n^2 - m_p^2 + m_e^2)c^2 - m_nc\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}}{\left(m_nc - |\mathbf{p}_4| - \sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}\right)}.$$

This is the upper limit on the $|\mathbf{p}_2|$ integral.
Similarly,

$$\begin{aligned}
u_+^2 &= \cancel{|\mathbf{p}_2|^2} + \cancel{|\mathbf{p}_4|^2} + 2|\mathbf{p}_2||\mathbf{p}_4| + m_p^2c^2 > \left(m_nc - |\mathbf{p}_2| - \sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}\right)^2 \\
&= m_n^2c^2 + \cancel{|\mathbf{p}_2|^2} + \cancel{|\mathbf{p}_4|^2} + m_e^2c^2 - 2m_nc|\mathbf{p}_2| - 2m_nc\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2} \\
&\quad + 2|\mathbf{p}_2|\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2};
\end{aligned}$$

$$2|\mathbf{p}_2|\left(m_nc + |\mathbf{p}_4| - \sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}\right) > (m_n^2 - m_p^2 + m_e^2)c^2 - 2m_nc\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2};$$

$$|\mathbf{p}_2| > \frac{\frac{1}{2}(m_n^2 - m_p^2 + m_e^2)c^2 - m_nc\sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}}{\left(m_nc + |\mathbf{p}_4| - \sqrt{|\mathbf{p}_4|^2 + m_e^2c^2}\right)}.$$

This is the lower limit on the $|\mathbf{p}_2|$ integral.

(b) Start with Eq. 9.52, and eliminate m_e (in favor of δm_n), $|\mathbf{p}_4|$ (in favor of $\phi m_n c$), and m_p (in favor of $(1-\epsilon)m_n$):

$$\begin{aligned}
p_\pm &= \frac{\frac{1}{2}\left[m_n^2-(1-\epsilon)^2 m_n^2+\delta^2 m_n^2\right]c^2 - m_n c\sqrt{\phi^2 m_n^2 c^2+\delta^2 m_n^2 c^2}}{m_n c - \sqrt{\phi^2 m_n^2 c^2+\delta^2 m_n^2 c^2} \mp \phi m_n c} \\
&= m_n c\left[\frac{\frac{1}{2}(1-1+2\epsilon-\epsilon^2+\delta^2)-\sqrt{\phi^2+\delta^2}}{1-\sqrt{\phi^2+\delta^2}\mp\phi}\right] = m_n c\left[\frac{\epsilon-\eta+(\delta^2-\epsilon^2)/2}{1-\eta\mp\phi}\right] \\
&\approx m_n c(\epsilon-\eta)\left[1+\frac{(\delta^2-\epsilon^2)}{2(\epsilon-\eta)}\right][1+(\eta\pm\phi)] \\
&\approx m_n c(\epsilon-\eta)\left[1+\eta\mp\phi+\frac{(\delta^2-\epsilon^2)}{2(\epsilon-\eta)}\right].
\end{aligned}$$

Expanding to lowest order in the small quantities,

$$\begin{aligned}
p_+ - p_- &= -2m_n c(\epsilon-\eta)\phi, \\
p_+ + p_- &= 2m_n c(\epsilon-\eta), \\
p_+^2 - p_-^2 &= (p_+-p_-)(p_++p_-) = -4m_n^2c^2(\epsilon-\eta)^2\phi, \\
p_+^2 + p_+p_- + p_-^2 &= 3m_n^2c^2(\epsilon-\eta)^2, \\
p_+^3 - p_-^3 &= (p_+-p_-)(p_+^2+p_+p_-+p_-^2) = -6m_n^3c^3(\epsilon-\eta)^3\phi.
\end{aligned}$$

So

$$\begin{aligned}
J &\approx \tfrac{1}{2}\left[m_n^2-(1-\epsilon)^2m_n^2-\delta^2m_n^2\right]c^4\left[-4m_n^2c^2(\epsilon-\eta)^2\phi\right] \\
&\quad -\tfrac{2}{3}m_n c^3\left[-6m_n^3c^3(\epsilon-\eta)^3\phi\right] \\
&= 2m_n^4c^6(\epsilon-\eta)^2\phi(2\epsilon-2\eta-2\epsilon) = -4m_n^4c^6(\epsilon-\eta)^2\phi\eta.
\end{aligned}$$

Problem 9.10

This is a special case of Problem 3.22(a), with $m_D = 0$ (the neutrino). So

$$E_{\max} = \frac{m_n^2+m_e^2-m_p^2}{2m_n}c^2.$$

The electron comes out one direction, the proton diametrically opposite, and the neutrino has zero energy and zero momentum (compare Problem 3.19).

Numerically,

$$(m_n-m_p)c^2 = 1.2933318 \text{ MeV}$$

$$m_e = 0.510998902\text{ MeV},\quad m_n = 939.56533\text{ MeV},\quad m_p = 938.27200\text{ MeV}$$

$$E_{\max} = \frac{1.2933318(939.56533 + 938.27200) + (0.510998902)^2}{2(939.56533)} = 1.2925806\,\text{MeV}$$

$$\frac{1.2933318 - 1.2925806}{1.2925806} = 5.812 \times 1-^{-4}, \quad \text{or} \quad \boxed{0.058\%}.$$

Problem 9.11

(a) Let $E_1 \equiv m_e c^2$ and $E_2 \equiv (m_n - m_p)c^2$; the integral is

$$\begin{aligned}
I &\equiv \int_{E_1}^{E_2} E\sqrt{E^2 - E_1^2}(E_2 - E)^2\,dE \\
&= E_2^2 \int_{E_1}^{E_2} E\sqrt{E^2 - E_1^2}\,dE - 2E_2 \int_{E_1}^{E_2} E^2\sqrt{E^2 - E_1^2}\,dE + \int_{E_1}^{E_2} E^3\sqrt{E^2 - E_1^2}\,dE.
\end{aligned}$$

From integral tables,

$$\begin{aligned}
\int_{E_1}^{E_2} E\sqrt{E^2 - E_1^2}\,dE &= \frac{1}{3}(E^2 - E_1^2)^{3/2}\Big|_{E_1}^{E_2} = \frac{1}{3}(E_2^2 - E_1^2)^{3/2}; \\
\int_{E_1}^{E_2} E^2\sqrt{E^2 - E_1^2}\,dE &= \left[\frac{E}{4}(E^2 - E_1^2)^{3/2} + \frac{E_1^2}{8}E\sqrt{E^2 - E_1^2} - \frac{E_1^4}{8}\ln\left(E + \sqrt{E^2 - E_1^2}\right)\right]\Big|_{E_1}^{E_2} \\
&= \frac{E_2}{4}(E_2^2 - E_1^2)^{3/2} + \frac{E_1^2 E_2}{8}\sqrt{E_2^2 - E_1^2} - \frac{E_1^4}{8}\ln\left(\frac{E_2 + \sqrt{E_2^2 - E_1^2}}{E_1}\right); \\
\int_{E_1}^{E_2} E^3\sqrt{E^2 - E_1^2}\,dE &= \frac{1}{5}\left(E^2 - E_1^2\right)^{5/2} + \frac{E_1^2}{3}\left(E^2 - E_1^2\right)^{3/2}\Big|_{E_1}^{E_2} \\
&= \frac{1}{5}\left(E_2^2 - E_1^2\right)^{5/2} + \frac{E_1^2}{3}\left(E_2^2 - E_1^2\right)^{3/2}.
\end{aligned}$$

So

$$\begin{aligned}
I &= E_2^2\frac{1}{3}(E_2^2 - E_1^2)^{3/2} \\
&\quad - 2E_2\left[\frac{E_2}{4}(E_2^2 - E_1^2)^{3/2} + \frac{E_1^2 E_2}{8}\sqrt{E_2^2 - E_1^2} - \frac{E_1^4}{8}\ln\left(\frac{E_2 + \sqrt{E_2^2 - E_1^2}}{E_1}\right)\right] \\
&\quad + \frac{1}{5}\left(E_2^2 - E_1^2\right)^{5/2} + \frac{E_1^2}{3}\left(E_2^2 - E_1^2\right)^{3/2}, \\
&= \sqrt{E_2^2 - E_1^2}\Big[\frac{1}{3}E_2^2(E_2^2 - E_1^2) - \frac{1}{2}E_2^2(E_2^2 - E_1^2) - \frac{1}{4}E_1^2 E_2^2 + \frac{1}{5}(E_2^2 - E_1^2)^2 \\
&\quad + \frac{1}{3}E_1^2(E_2^2 - E_1^2)\Big] - \frac{1}{4}E_1^4 E_2\ln\left(\frac{E_2 + \sqrt{E_2^2 - E_1^2}}{E_1}\right)
\end{aligned}$$

$$I = \frac{1}{60}\sqrt{E_2^2 - E_1^2}\left(2E_2^4 - 9E_1^2E_2^2 - 8E_1^2\right) - \frac{1}{4}E_1^4E_2 \ln\left(\frac{E_2 + \sqrt{E_2^2 - E_1^2}}{E_1}\right)$$
$$= \frac{1}{4}E_1^5\left[\frac{1}{15}(2a^4 - 9a^2 - 8)\sqrt{a^2-1} - a\ln(a + \sqrt{a^2-1})\right],$$

where $a \equiv E_2/E_1 = (m_n - m_p)/m_e$. So the integral of Eq. 9.59 is

$$\Gamma = \frac{1}{\pi^3\hbar}\left(\frac{g_w}{2M_Wc^2}\right)^4 \frac{1}{4}(m_ec^2)^5\left[\frac{1}{15}(2a^4 - 9a^2 - 8)\sqrt{a^2-1} - a\ln(a + \sqrt{a^2-1})\right].$$

(b) If $a \gg 1$ (not a very good approximation for the neutron, since in fact $a = 1.2933318/0.510998902 = 2.530987$, but it would be valid for other semileptonic decays) then the first term dominates, and

$$\Gamma = \frac{1}{30\pi^3\hbar}\left(\frac{g_w}{2M_Wc^2}\right)^4 (\Delta mc^2)^5.$$

Problem 9.12

Putting in the empirical value $a = 2.530987$, the term in square brackets in Eq. 9.60 evaluates to 6.54438, so

$$\Gamma = \frac{6.54438}{4\pi^3(6.58212 \times 10^{-22})}\left(\frac{0.653}{2(80423)}\right)^4 (0.5109989)^5 = 7.5876 \times 10^{-4}\ /\mathrm{s},$$

and hence

$$\tau = 1318\ \mathrm{s}.$$

Problem 9.13

From the Feynman diagram for neutron decay, $\mathbf{p}_W = \mathbf{p}_n - \mathbf{p}_p$, so (for a neutron at rest) the maximum W momentum is the same as the maximum proton momentum (in magnitude). For the same reasoning as in Problem 9.10, this occurs in the limit when the neutrino energy is zero, so it's essentially a two-body decay ($n \to p + e$), and hence the proton momentum is given by Problem 3.19(b):

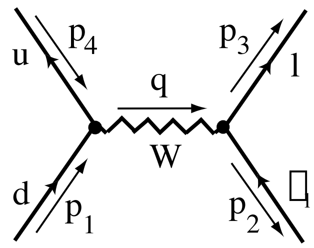
$$\frac{\sqrt{(m_n + m_p + m_e)c^2(m_n + m_p - m_e)c^2(m_n - m_p + m_e)c^2(m_n - m_p - m_e)c^2}}{2(m_nc^2)c}$$
$$= \frac{\sqrt{(1877.9 + 0.5)(1877.9 - 0.5)(1.2933 + 0.5110)(1.2933 - 0.5110)}}{2(939.6)c}$$
$$= 1.187\ \mathrm{MeV}/c.$$

$$\lambda = \frac{h}{|\mathbf{p}|} = \frac{2\pi(6.582 \times 10^{-22})(2.998 \times 10^{8})}{1.187} = \boxed{1.04 \times 10^{-12}\ \mathrm{m}}.$$

This is about a thousand times the diameter of a neutron. So the W is insensitive to the internal structure of the neutron (and the proton), which is why we were able to treat them as elementary particles in calculating the neutron lifetime.

Problem 9.14

The Feynman diagram for this process, $\pi^-(d + \bar{u}) \to \ell + \bar{\nu}_\ell$, is



Calculating the amplitude:

$$\int \left[\bar{v}(4)\cos\theta_C \frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)u(1)\right] \frac{ig_{\mu\nu}}{(M_W c)^2} \left[\bar{u}(3)\frac{-ig_w}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)v(2)\right]$$
$$\times(2\pi)^4\delta^4(p_1+p_4-q)(2\pi)^4\delta^4(q-p_2-p_3)\frac{d^4q}{(2\pi)^4}.$$

Use the first delta function to do the integral ($q \to p_1 + p_4$), cancel $(2\pi)^4\delta^4(p_1 + p_4 - p_2 - p_3)$, and multiply by i:

$$\mathcal{M} = \frac{g_w^2 \cos\theta_C}{8(M_W c)^2}\left[\bar{v}(4)\gamma^\mu(1-\gamma^5)u(1)\right]\left[\bar{u}(3)\gamma_\mu(1-\gamma^5)v(2)\right],$$

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\left[\frac{g_w^2\cos\theta_C}{8(M_W c)^2}\right]^2 \mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_1 c)\gamma^\nu(1-\gamma^5)(\not{p}_4 - m_4 c)\right]$$
$$\times\mathrm{Tr}\left[\gamma_\mu(1-\gamma^5)(\not{p}_2 - m_2 c)\gamma_\nu(1-\gamma^5)(\not{p}_3 + m_3 c)\right]$$

Here $m_1 = m_4 = m$, $m_2 = 0$, and $m_3 = m_\ell$ (but the term in m_3 involves an odd number of gamma matrices, so it drops out).

$$\langle|\mathcal{M}|^2\rangle = \cos^2\theta_C\left(\frac{g_w}{4M_W c}\right)^4 \mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)(\not{p}_1 + mc)\gamma^\nu(1-\gamma^5)(\not{p}_4 - mc)\right]$$
$$\times\mathrm{Tr}\left[\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)\not{p}_3\right].$$

These traces were evaluated in Problem 9.2; quoting the result (Eq. 9.159), with $c_V = c_A = 1$,

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{\cos^2\theta_C}{4}\left(\frac{g_w}{M_W c}\right)^4 \left\{\left[p_1^\mu p_4^\nu + p_1^\nu p_4^\mu - g^{\mu\nu}(p_1\cdot p_4)\right] - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{4\sigma}\right\} \\
&\qquad\times\left\{\left[p_{2\mu}p_{3\nu} + p_{2\nu}p_{3\mu} - g_{\mu\nu}(p_2\cdot p_3)\right] - i\epsilon_{\mu\nu\kappa\tau}p_2^\kappa p_3^\tau\right\} \\
&= \frac{\cos^2\theta_C}{4}\left(\frac{g_w}{M_W c}\right)^4 \left\{-\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\kappa\tau}p_{1\lambda}p_{4\sigma}p_2^\kappa p_3^\tau\right. \\
&\quad + \left.\left[p_1^\mu p_4^\nu + p_1^\nu p_4^\mu - g^{\mu\nu}(p_1\cdot p_4)\right]\left[p_{2\mu}p_{3\nu} + p_{2\nu}p_{3\mu} - g_{\mu\nu}(p_2\cdot p_3)\right]\right\} \\
&= \frac{\cos^2\theta_C}{4}\left(\frac{g_w}{M_W c}\right)^4 \left[2\left(\delta_\kappa^\lambda\delta_\tau^\sigma - \delta_\tau^\lambda\delta_\kappa^\sigma\right)p_{1\lambda}p_{4\sigma}p_2^\kappa p_3^\tau\right. \\
&\qquad\left. + 2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_3)(p_2\cdot p_4)\right] \\
&= \cos^2\theta_C\left(\frac{g_w}{M_W c}\right)^4 (p_1\cdot p_2)(p_3\cdot p_4).
\end{aligned}$$

In the CM frame (pion at rest),

$$p_1 = (E_1/c, \mathbf{p}_i), \quad p_4 = (E_4/c, -\mathbf{p}_i), \quad p_3 = (E_3/c, \mathbf{p}_f), \quad p_2 = (E_2/c, -\mathbf{p}_f),$$

where $E_1 = E_4 = m_\pi c^2/2$ and the outgoing energies and momenta are given by the formulas in Problem 3.19:

$$E_2 = \frac{m_\pi^2 - m_\ell^2}{2m_\pi}c^2, \quad E_3 = \frac{m_\pi^2 + m_\ell^2}{2m_\pi}c^2, \quad |\mathbf{p}_f| = \frac{m_\pi^2 - m_\ell^2}{2m_\pi}c.$$

Finally,

$$E_1^2 - \mathbf{p}_i^2 c^2 = m^2 c^4 \;\Rightarrow\; \mathbf{p}_i^2 = \frac{m_\pi^2 c^2}{4} - m^2 c^2 \approx \left(\frac{m_\pi c}{2}\right)^2$$

(the quarks are effectively massless, compared to the pion), so $|\mathbf{p}_i| = m_\pi c/2$.

$$\begin{aligned}
p_1\cdot p_2 &= \frac{E_1 E_2}{c^2} - \mathbf{p}_1\cdot\mathbf{p}_2 = (m_\pi^2 - m_\ell^2)\frac{c^2}{4} + \mathbf{p}_i\cdot\mathbf{p}_f; \\
p_3\cdot p_4 &= \frac{E_3 E_4}{c^2} - \mathbf{p}_3\cdot\mathbf{p}_4 = (m_\pi^2 + m_\ell^2)\frac{c^2}{4} + \mathbf{p}_i\cdot\mathbf{p}_f; \\
\mathbf{p}_i\cdot\mathbf{p}_f &= |\mathbf{p}_i||\mathbf{p}_f|\cos\theta = (m_\pi^2 - m_\ell^2)\frac{c^2}{4}\cos\theta.
\end{aligned}$$

So

$$p_1\cdot p_2 = (m_\pi^2 - m_\ell^2)\frac{c^2}{4}(1+\cos\theta), \; p_3\cdot p_4 = \left[(m_\pi^2 + m_\ell^2) + (m_\pi^2 - m_\ell^2)\cos\theta\right]\frac{c^2}{4}$$

and $\langle|\mathcal{M}|^2\rangle$ is

$$\cos^2\theta_C\left(\frac{g_w}{2M_W}\right)^4 (m_\pi^2 - m_\ell^2)(1+\cos\theta)\left[(m_\pi^2 + m_\ell^2) + (m_\pi^2 - m_\ell^2)\cos\theta\right].$$

The differential scattering cross-section is (Eq. 6.47):

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\langle|\mathcal{M}|^2\rangle}{(m_\pi c^2)^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \\
&= \left(\frac{\hbar}{8\pi m_\pi c}\right)^2 \cos^2\theta_C \left(\frac{g_w}{2M_W}\right)^4 (m_\pi^2 - m_\ell^2)(1+\cos\theta) \\
&\qquad \times \left[(m_\pi^2 + m_\ell^2) + (m_\pi^2 - m_\ell^2)\cos\theta\right]\frac{(m_\pi^2 - m_\ell^2)}{m_\pi^2} \\
&= \left[\frac{\hbar g_w^2 \cos\theta_C}{32\pi m_\pi^2 M_W^2 c}(m_\pi^2 - m_\ell^2)\right]^2 (1+\cos\theta)\left[(m_\pi^2 + m_\ell^2) + (m_\pi^2 - m_\ell^2)\cos\theta\right]
\end{aligned}$$

and the total cross-section is

$$\begin{aligned}
\sigma &= \left[\frac{\hbar g_w^2 \cos\theta_C}{32\pi m_\pi^2 M_W^2 c}(m_\pi^2 - m_\ell^2)\right]^2 2\pi \\
&\qquad \times \int_0^\pi (1+\cos\theta)\left[(m_\pi^2 + m_\ell^2) + (m_\pi^2 - m_\ell^2)\cos\theta\right]\sin\theta\, d\theta
\end{aligned}$$

The integral consists of three terms:

$$\begin{aligned}
&(m_\pi^2 + m_\ell^2)\int_0^\pi \sin\theta\, d\theta + 2m_\pi^2 \int_0^\pi \cos\theta\sin\theta\, d\theta \\
&\quad + (m_\pi^2 - m_\ell^2)\int_0^\pi \cos^2\theta\sin\theta\, d\theta \\
&= 2(m_\pi^2 + m_\ell^2) + \tfrac{2}{3}(m_\pi^2 - m_\ell^2) = \tfrac{4}{3}(2m_\pi^2 + m_\ell^2).
\end{aligned}$$

Thus

$$\sigma = \frac{2}{3\pi}\left[\frac{\hbar g_w^2 \cos\theta_C}{16 m_\pi^2 M_W^2 c}(m_\pi^2 - m_\ell^2)\right]^2 \left(2m_\pi^2 + m_\ell^2\right).$$

From Eq. 7.171, the pion decay rate is given by

$$\Gamma = v\sigma|\psi(0)|^2,$$

and $v \approx c$, since we are treating the quarks as essentially massless. Comparing Eq. 9.76, we conclude that

$$f_\pi^2 = \boxed{\frac{2\hbar^3}{3c}\frac{(2m_\pi^2 + m_\ell^2)}{m_\pi m_\ell^2}\cos^2\theta_C|\psi(0)|^2}.$$

Problem 9.15

If $E \gg mc^2$, then $E + mc^2 \cong E$. Also, since $E^2 - \mathbf{p}^2c^2 = m^2c^4$, $E \cong |\mathbf{p}|c$, so

$$\frac{c(\mathbf{p}\cdot\boldsymbol{\sigma})}{E+mc^2} \cong \frac{(\mathbf{p}\cdot\boldsymbol{\sigma})}{(E/c)} \cong \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{|\mathbf{p}|} = (\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}).$$

$$\gamma^5 u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_A \\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})u_A \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})u_A \\ u_A \end{pmatrix}$$

But

$$\begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) \end{pmatrix} u \cong \begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) \end{pmatrix} \begin{pmatrix} u_A \\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})u_A \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})u_A \\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 u_A \end{pmatrix}.$$

However [Problem 4.20(c)], $(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 = (\hat{\mathbf{p}})^2 = 1$, so

$$\gamma^5 u \cong \begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) \end{pmatrix} u. \checkmark$$

Now

$$\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \cdot \hat{\mathbf{p}} = \begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}, \quad \text{so}$$

$$\begin{aligned}(\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}})(P_\pm u) &= \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}(1 \pm \gamma^5)u \\ &= \frac{1}{2}\Bigg[\underbrace{\begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) \end{pmatrix} u}_{\cong\gamma^5 u} \pm \underbrace{\begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) \end{pmatrix}\gamma^5 u}_{\bigstar}\Bigg]\end{aligned}$$

where

$$\bigstar = \begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) \end{pmatrix}^2 u = \begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 \end{pmatrix} u = u.$$

So

$$(\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}})(P_\pm u) = \frac{1}{2}(\gamma^5 u \pm u) = \pm\frac{1}{2}(1 \pm \gamma^5)u = \pm P_\pm u. \checkmark$$

Problem 9.16

The formula is the same as for π^- decay (Eq. 9.77), with $m_\pi \longrightarrow m_K$:

$$\begin{aligned}\frac{\Gamma(K^- \longrightarrow e^- + \bar{\nu}_e)}{\Gamma(K^- \longrightarrow \mu^- + \bar{\nu}_\mu)} &= \frac{m_e^2(m_K^2 - m_e^2)^2}{m_\mu^2(m_K^2 - m_\mu^2)^2} \\ &= \left[\frac{(0.510999)\{(493.667)^2 - (0.510999)^2\}}{(105.6584)\{(493.667)^2 - (105.6584)^2\}}\right]^2 = \boxed{2.57 \times 10^{-5}}.\end{aligned}$$

The experimental ratio (using data from the *Particle Physics Booklet*) is

$$\frac{1.55 \times 10^{-5}}{0.6344} = \boxed{2.44 \times 10^{-5}}.$$

The agreement is quite good.

Problem 9.17

From Example 9.3, we have

$$\begin{aligned}\Gamma &= \frac{1}{30\pi^3\hbar}\left(\frac{g_w}{2M_W c^2}\right)^4 (\Delta mc^2)^5 X^2 \\ &= \frac{1}{30\pi^3(6.58212 \times 10^{-22})}\left(\frac{0.653}{2(80423)}\right)^4 (\Delta mc^2)^5 X^2 \\ &= (1.4123 \times 10^{-4})(\Delta mc^2)^5 X^2.\end{aligned}$$

(a)

$$\begin{aligned}\Sigma^0(uds) &\rightarrow \Sigma^+(uus) + e + \bar{\nu}_e : \ \Delta mc^2 = (1192.642) - (1189.37) = 3.372. \\ d \rightarrow u : \Sigma^0 &= [(us - su)d + (ds - sd)u]/2 \rightarrow [(us - su)u + (us - su)u]/2 \\ &= (us - su)u = \sqrt{2}\,\Sigma^+, \quad \text{so} \quad X = \sqrt{2}\cos\theta_C = 1.377. \\ \Gamma &= (1.4123 \times 10^{-4})(3.372)^5(1.377)^2 = \boxed{0.1167/\text{s}}.\end{aligned}$$

(b)

$$\begin{aligned}\Sigma^-(dds) &\rightarrow \Lambda(uds) + e + \bar{\nu}_e : \ \Delta mc^2 = 81.766. \\ d \rightarrow u : \Sigma^- &= (ds - sd)d/\sqrt{2} \rightarrow (us - su)d/\sqrt{2} + (ds - sd)u/\sqrt{2} \\ &= \sqrt{2}\Sigma^0, \quad \text{so} \quad X = 0. \\ \Gamma &= \boxed{0}.\end{aligned}$$

(c)

$$\begin{aligned}\Xi^-(dss) &\rightarrow \Xi^0(uss) + e + \bar{\nu}_e : \ \Delta mc^2 = 6.48. \\ d \rightarrow u : \Xi^- &= [(ds - sd)s]/\sqrt{2} \rightarrow [(us - su)s]/\sqrt{2} \\ &= \Xi^0, \quad \text{so} \quad X = \cos\theta_C = 0.9738. \\ \Gamma &= (1.4123 \times 10^{-4})(6.48)^5(0.9738)^2 = \boxed{1.530/\text{s}}.\end{aligned}$$

(d)

$$\begin{aligned}
\Lambda(uds) &\rightarrow p(uud) + e + \bar{\nu}_e: \ \Delta mc^2 = 1115.683 - 938.272 = 177.4. \\
s \rightarrow u: \Lambda &= [2(ud - du)s + (us - su)d - (ds - sd)u]/\sqrt{12} \\
&\rightarrow [2(ud - du)u + (uu - uu)d - (du - ud)u]/\sqrt{12} \\
&= 3(ud - du)u/\sqrt{12} = \sqrt{3/2}\, p, \quad \text{so} \quad X = \sqrt{3/2}\sin\theta_C = 0.2786. \\
\Gamma &= (1.4123 \times 10^{-4})(177.4)^5(0.2786)^2 = \boxed{1.926 \times 10^6/\text{s}}.
\end{aligned}$$

(e)

$$\begin{aligned}
\Sigma^-(dds) &\rightarrow n(udd) + e + \bar{\nu}_e: \ \Delta mc^2 = (1197.449) - (939.565) = 257.9. \\
s \rightarrow u: \Sigma^- &= [(ds - sd)d]/\sqrt{2} \rightarrow [(du - ud)d]/\sqrt{2} \\
&= -n, \quad \text{so} \quad X = -\sin\theta_C = -0.2275. \\
\Gamma &= (1.4123 \times 10^{-4})(257.9)^5(0.2275)^2 = \boxed{8.340 \times 10^6/\text{s}}.
\end{aligned}$$

(f)

$$\begin{aligned}
\Xi^0(uss) &\rightarrow \Sigma^+(uus) + e + \bar{\nu}_e: \ \Delta mc^2 = (1314.83) - (1189.37) = 125.5. \\
s \rightarrow u: \Xi^0 &= [(us - su)s]/\sqrt{2} \rightarrow [(uu - uu)s + (us - su)u]/\sqrt{2} \\
&= [(us - su)u]/\sqrt{2} = \Sigma^+, \quad \text{so} \quad X = \sin\theta_C = 0.2275. \\
\Gamma &= (1.4123 \times 10^{-4})(125.5)^5(0.2275)^2 = \boxed{2.276 \times 10^5/\text{s}}.
\end{aligned}$$

To compare the experimental rates, note that the branching ratio is $\Gamma/\Gamma_{\text{tot}} = \Gamma\tau$, so $\Gamma = (b.r.)/\tau$.
(a) Unknown.
(b) $(5.73 \times 10^{-5})/(1.479 \times 10^{-10}) = 3.87 \times 10^5/\text{s}$.
(c) $< (2.3 \times 10^{-3})/(1.639 \times 10^{-10}) = 1.4 \times 10^7/\text{s}$.
(d) $(8.32 \times 10^{-4})/(2.631 \times 10^{-10}) = 3.16 \times 10^6/\text{s}$.
(e) $(1.017 \times 10^{-3})/(1.479 \times 10^{-10}) = 6.88 \times 10^6/\text{s}$.
(f) $(2.7 \times 10^{-4})/(2.90 \times 10^{-10}) = 9.3 \times 10^5/\text{s}$.
The agreement is not great, but this is because we are ignoring corrections to the axial vector coupling (corrections that could be calculated in the full Cabibbo model). (a) has not been measured, (b) is evidently due entirely to axial coupling corrections, and (c) is only a lower bound—so there is no real test in these. (d) is off by 1/3, (e) is off by 1/7 (not too bad), but for (f) the calculated value is less than a third of the measured number.

Problem 9.18

(a) Referring to Figures 9.4 and 9.5, and the same for t (in place of u or c), and so on (for putative 4th and higher generations), and Eq. 9.81, the total amplitude is proportional to

$$[V_{ud}^* V_{us} + V_{cd}^* V_{cs} + V_{td}^* V_{ts} + \cdots] = \sum_{j=1}^{n} V_{jd}^* V_{js} = \sum_{j=1}^{n} \tilde{V}_{dj}^* V_{js} = (V^\dagger V)_{ds}.$$

But if V is unitary, then $(V^\dagger V)_{ds} = \delta_{ds} = 0$. QED

(b) Since $H = H^\dagger = \tilde{H}^*$, the diagonal elements are real ($H_{ii}^* = H_{ii} \Rightarrow n$ real numbers), and the off-diagonal elements are complex conjugate pairs ($H_{ij}^* = H_{ji} \Rightarrow$ two real numbers for each element above the main diagonal—which is to say, one real number for every off-diagonal element, of which there are $n^2 - n$). So it takes a total of n^2 real numbers to characterize an $n \times n$ Hermitian matrix, and hence also $\boxed{n^2}$ real numbers to characterize an $n \times n$ unitary matrix. In particular, there are $\boxed{9}$ real parameters in a 3×3 unitary matrix.

If we write, for instance, $V_{us} = \langle u|V|s\rangle$, it is clear that changing the (arbitrary) phase of the s quark wave function ($|s\rangle \to e^{i\theta}|s\rangle$) will change V_{us}, V_{cs}, and V_{ts} by this same factor, whereas changing the phase of the u quark wave function ($|u\rangle \to e^{i\phi}|u\rangle$) will change V_{ud}, V_{us}, and V_{ub} by the factor $e^{-i\phi}$. On the other hand, changing *all* the quark phases by the *same* amount will not affect the elements of the CKM matrix. So there are $2n - 1$ arbitrary phase factors in the CKM matrix, which we can choose so as to make the matrix as real as possible.

(c) A real orthogonal matrix is unitary ($O = O^*$, $O\tilde{O} = 1 \Rightarrow O\tilde{O}^* = OO^\dagger = 1$), so it can be written in the form $O = e^{iA}$, where A is Hermitian. But $O = O^* \Rightarrow e^{iA} = e^{-iA^*} \Rightarrow A = -A^*$ (so A is imaginary). But $A = A^\dagger = \tilde{A}^* = -\tilde{A}$, so A is antisymmetric. Its diagonal elements are all zero, and its diagonal elements are imaginary and opposite in pairs ($A_{ij} = -A_{ji}$), so the question is how many elements there are above the main diagonal. *Answer:* half the total number of off-diagonal elements, which is to say, $\boxed{(1/2)(n^2 - n)}$. In particular, there are $\boxed{3}$ real parameters in a 3×3 (real) orthogonal matrix.

(d) Because the CKM matrix is unitary, it contains n^2 real parameters, but $2n - 1$ of these can be eliminated by appropriate choice of the quark phases, leaving $n^2 - (2n - 1) = (n - 1)^2$. In particular, for the 3×3 case, there are 4 real parameters. But a 3 *real* unitary (which is to say, orthogonal) matrix carries only 3 real parameters. So you $\boxed{\textit{cannot}}$ reduce the general 3×3 CKM matrix to a real matrix. But for $n = 2$ the CKM matrix would have 1 real parameter, and a real orthogonal 2×2 matrix has $(1/2)(4 - 2) = 1$, so with only two generations you $\boxed{\textit{could}}$ reduce the CKM matrix to real form. So if CP violation comes from an imaginary term in the CKM matrix, it cannot occur unless there are (at least) three generations.

Problem 9.19

$$VV^\dagger = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$$\times \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta} & s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta} \\ s_{12}c_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta} \\ s_{13}e^{i\delta} & s_{23}c_{13} & c_{23}c_{13} \end{pmatrix}$$

$$\begin{aligned}
(VV^\dagger)_{11} &= c_{12}c_{13}c_{12}c_{13} + s_{12}c_{13}s_{12}c_{13} + s_{13}e^{-i\delta}s_{13}e^{i\delta} \\
&= \cos^2\theta_{13}(\cos^2\theta_{12} + \sin^2\theta_{12}) + \sin^2\theta_{13} = \cos^2\theta_{13} + \sin^2\theta_{13} = 1. \\
(VV^\dagger)_{12} &= c_{12}c_{13}[\cancel{-s_{12}c_{23}} - c_{12}s_{23}s_{13}e^{-i\delta}] + s_{12}c_{13}[\cancel{c_{12}c_{23}} - s_{12}s_{23}s_{13}e^{-i\delta}] \\
&\quad + s_{13}e^{-i\delta}s_{23}c_{13} = c_{13}s_{13}s_{23}e^{-i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0. \\
(VV^\dagger)_{13} &= c_{12}c_{13}[\cancel{s_{12}s_{23}} - c_{12}c_{23}s_{13}e^{-i\delta}] + s_{12}c_{13}[\cancel{-c_{12}s_{23}} - s_{12}c_{23}s_{13}e^{-i\delta}] \\
&\quad + s_{13}e^{-i\delta}c_{23}c_{13} = c_{13}s_{13}c_{23}e^{-i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0. \\
(VV^\dagger)_{21} &= [\cancel{-s_{12}c_{23}} - c_{12}s_{23}s_{13}e^{i\delta}]c_{12}c_{13} + [\cancel{c_{12}c_{23}} - s_{12}s_{23}s_{13}e^{i\delta}]s_{12}c_{13} \\
&\quad + s_{23}c_{13}s_{13}e^{i\delta} = s_{23}c_{13}s_{13}e^{i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0. \\
(VV^\dagger)_{22} &= [-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}][-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta}] \\
&\quad + [c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}][c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta}] + s_{23}c_{13}s_{23}c_{13} \\
&= s_{12}^2c_{23}^2 + c_{12}^2s_{23}^2s_{13}^2 + c_{12}^2c_{23}^2 + s_{12}^2s_{23}^2s_{13}^2 + s_{23}^2c_{13}^2 \\
&= c_{23}^2 + s_{23}^2s_{13}^2 + s_{23}^2c_{13}^2 = 1. \\
(VV^\dagger)_{23} &= [-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}][s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta}] \\
&\quad + [c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}][-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta}] + s_{23}c_{13}c_{23}c_{13} \\
&= -s_{12}^2s_{23}c_{23} + c_{12}^2s_{12}^2s_{23}c_{23} - c_{12}^2s_{23}c_{23} + s_{12}^2s_{13}^2s_{23}c_{23} + c_{13}^2s_{23}c_{23} \\
&= s_{23}c_{23}[-1 + s_{13}^2 + c_{13}^2] = 0. \\
(VV^\dagger)_{31} &= [\cancel{s_{12}s_{23}} - c_{12}c_{23}s_{13}e^{i\delta}]c_{12}c_{13} + [\cancel{-c_{12}s_{23}} - s_{12}c_{23}s_{13}e^{i\delta}]s_{12}c_{13} \\
&\quad + c_{23}c_{13}s_{13}e^{i\delta} = c_{23}c_{13}s_{13}e^{i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0. \\
(VV^\dagger)_{32} &= [s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}][-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta}] \\
&\quad + [-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta}][c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta}] + c_{23}c_{13}s_{23}c_{13} \\
&= -s_{12}^2s_{23}c_{23} + c_{12}^2s_{13}^2s_{23}c_{23} - c_{12}^2s_{23}c_{23} + s_{12}^2s_{13}^2s_{23}c_{23} + c_{13}^2s_{23}c_{23} \\
&= s_{23}c_{23}[-1 + s_{13}^2 + c_{13}^2] = 0. \\
(VV^\dagger)_{33} &= [s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}][s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta}] \\
&\quad + [-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta}][-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta}] + c_{23}c_{13}c_{23}c_{13} \\
&= s_{12}^2s_{23}^2 + c_{12}^2c_{23}^2s_{13}^2 + c_{12}^2s_{23}^2 + s_{12}^2c_{23}^2s_{13}^2 + c_{23}^2c_{13}^2 \\
&= s_{23}^2 + c_{23}^2s_{13}^2 + c_{23}^2c_{13}^2 = 1.
\end{aligned}$$

So $VV^\dagger = 1$. ✓

Problem 9.20

According to Eq. 9.38,

$$\frac{G_F}{(\hbar c)^3} = \frac{\sqrt{2}}{8}\left(\frac{g_w}{M_W c^2}\right)^2 = 1.166 \times 10^{-5}/\text{GeV}^2,$$

so

$$M_W c^2 = \left(\frac{\sqrt{2}}{8}\frac{g_w^2 (\hbar c)^3}{G_F}\right)^{1/2}.$$

Equation 9.91 says

$$g_w = \frac{g_e}{\sin\theta_w}, \quad \text{where} \quad g_e = \sqrt{4\pi\alpha}$$

(QED Feynman rule #3). Using $\alpha = 1/137.036$ and $\sin^2\theta_w = 0.2314$ (Eq. 9.93):

$$\begin{aligned} M_W c^2 &= \left(\frac{\sqrt{2}}{8}\frac{4\pi\alpha}{\sin^2\theta_w (G_F/(\hbar c)^3}\right)^{1/2} \\ &= \left(\frac{\pi}{\sqrt{2}(137.036)(0.2314)(1.166\times 10^{-5})}\right)^{1/2} = \boxed{77.51\,\text{GeV}}. \end{aligned}$$

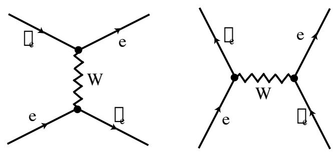
Equation 9.92 then yields

$$M_Z = \frac{M_W}{\cos\theta_w} = \frac{77.51}{\sqrt{1-0.2314}} = \boxed{88.41\,\text{Gev}}.$$

These predictions are not perfect ($M_W c^2 = 80.40$ GeV, $M_Z c^2 = 91.19$ GeV), but at the time they were a spectacular achievement. (You can do better using Eq. 9.39 for g_w, but that of course is circular, since M_W was used to *get* this value.)

Problem 9.21

With electron neutrinos (or antineutrinos) there are other diagrams, mediated by the W:



Problem 9.22

(a) The Feynman diagram is the same as Example 9.4, except that the arrows on the neutrino lines are reversed. Since we follow the fermion lines backwards through the diagram, Eq. 9.96 is replaced by

$$\mathscr{M} = \frac{g_z^2}{8(M_Z c)^2}[\bar{v}(1)\gamma^\mu(1-\gamma^5)v(3)][\bar{u}(4)\gamma_\mu(c_V - c_A\gamma^5)u(2)],$$

in effect switching $1 \leftrightarrow 3$. (This also switches the neutrino u to v's, and hence reverses the sign of the mass term in the competeness relation, but since these particles are massless, it doesn't matter.) Now, exchanging 1 and 3 in Eq. 9.97 is equivalent to $(c_V + c_A)^2 \leftrightarrow (c_V - c_A)^2$, which is to say, changing the sign of c_A. Making this change in Eqs. 9.99 and 9.100:

$$\frac{d\sigma}{d\Omega} = \boxed{2\left(\frac{\hbar c}{\pi}\right)^2\left(\frac{g_z}{4M_Z c^2}\right)^4 E^2\left[(c_V - c_A)^2 + (c_V + c_A)^2\cos^4\frac{\theta}{2}\right]},$$

$$\sigma = \boxed{\frac{2}{3\pi}(\hbar c)^2\left(\frac{g_z}{2M_Z c^2}\right)^4 E^2(c_V^2 + c_A^2 - c_V c_A)}.$$

(b) From (a) and Eq. 9.100,

$$R \equiv \frac{\sigma(\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-)}{\sigma(\nu_\mu + e^- \to \nu_\mu + e^-)} = \frac{c_V^2 + c_A^2 - c_V c_A}{c_V^2 + c_A^2 + c_V c_A}.$$

For the electron (Table 9.1),

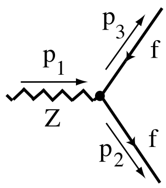
$$\begin{aligned} c_V^2 + c_A^2 &= \left(-\frac{1}{2} + 2\sin^2\theta_w\right)^2 + \frac{1}{4} = \frac{1}{2} - 2\sin^2\theta_w + 4\sin^2\theta_w \\ &= 0.5000 - 2(0.2314) + 4(0.2314)^2 = 0.2514; \\ c_V c_A &= \left(-\frac{1}{2} + 2\sin^2\theta_w\right)\left(-\frac{1}{2}\right) = \frac{1}{4} - \sin^2\theta_w = 0.2500 - 0.2314 = 0.0186. \end{aligned}$$

So

$$R = \frac{0.2514 - 0.0186}{0.2514 + 0.0186} = \boxed{0.862}.$$

Problem 9.23

(a)



Dropping the superscript f on c_V^f and c_A^f, for the moment,

$$\mathcal{M} = i\epsilon_\mu(1)\bar{u}(2)\left[-\frac{ig_z}{2}\gamma^\mu(c_V - c_A\gamma^5)\right]v(3)\left[\cancel{(2\pi)^4\delta^4(p_1 - p_2 - p_3)}\right];$$

$$|\mathcal{M}|^2 = \left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)\left[\bar{u}(2)\gamma^\mu(c_V - c_A\gamma^5)v(3)\right]\underbrace{\left[\bar{u}(2)\gamma^\nu(c_V - c_A\gamma^5)v(3)\right]^*}_{\bar{v}(3)\gamma^0[\gamma^\nu(c_V - c_A\gamma^5)]^\dagger\gamma^0 u(2)}$$

But

$$\gamma^0\left[\gamma^\nu(c_V - c_A\gamma^5)\right]^\dagger\gamma^0 = \gamma^0(c_V - c_A\gamma^5)^\dagger\gamma^{\nu\dagger}\gamma^0 = \gamma^0(c_V^* - c_A^*\gamma^5)\gamma^{\nu\dagger}\gamma^0$$

$$= (c_V^* + c_A^*\gamma^5)\gamma^0\gamma^{\nu\dagger}\gamma^0 = (c_V^* + c_A^*\gamma^5)\gamma^\nu = \gamma^\nu(c_V^* - c_A^*\gamma^5),$$

so

$$|\mathcal{M}|^2 = \left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)\left[\bar{u}(2)\gamma^\mu(c_V - c_A\gamma^5)v(3)\right]\left[\bar{v}(3)\gamma^\nu(c_V^* - c_A^*\gamma^5)u(2)\right].$$

Summing over the outgoing spins, we get

$$\left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)\mathrm{Tr}\left[\gamma^\mu(c_V - c_A\gamma^5)(\not{p}_3 - m_3c)\gamma^\nu(c_V^* - c_A^*\gamma^5)(\not{p}_2 + m_2c)\right].$$

The trace was calculated in Problem 9.2; quoting Eq. 9.159, with $m_2 = m_3 \to 0$,

$$\left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)4\Big\{\left(|c_V|^2 + |c_A|^2\right)\left[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - (p_2\cdot p_3)g^{\mu\nu}\right]$$
$$+ i\left(c_V c_A^* + c_V^* c_A\right)\epsilon^{\mu\nu\lambda\sigma}p_{2\lambda}p_{3\sigma}\Big\}.$$

The next step is to average over the (three) spin states of the Z, using the completeness relation in Problem 9.1 (Eq. 9.158):

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{3}g_z^2\left[-g_{\mu\nu} + \frac{p_{1\mu}p_{1\nu}}{(M_Zc)^2}\right](|c_V|^2 + |c_A|^2)\left[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - g^{\mu\nu}(p_2\cdot p_3)\right].$$

(The $\epsilon^{\mu\nu\lambda\sigma}$ term gives zero, since it is antisymmetric in $\mu \leftrightarrow \nu$, whereas $g_{\mu\nu}$ and $p_{1\mu}p_{1\nu}$ are symmetric.)

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{3}g_Z^2(|c_V|^2+|c_A|^2)\Big\{-2(p_2\cdot p_3)+4(p_2\cdot p_3)\\ &\quad+\frac{1}{(M_Zc)^2}\Big[2(p_1\cdot p_2)(p_1\cdot p_3)-\underbrace{(p_1\cdot p_1)}_{(M_Zc)^2}(p_2\cdot p_3)\Big]\Big\}\\ &= \frac{1}{3}g_z^2(|c_V|^2+|c_A|^2)\left[(p_2\cdot p_3)+2\frac{(p_1\cdot p_2)(p_1\cdot p_3)}{(M_Zc)^2}\right].\end{aligned}$$

The Golden Rule for 2-body decays in the CM frame (Eq. 6.35) says

$$\begin{aligned}\Gamma &= \frac{|\mathbf{p}|}{8\pi\hbar M_Z^2c}\langle|\mathcal{M}|^2\rangle\\ &= \frac{|\mathbf{p}|g_z^2}{24\pi\hbar M_Z^2c}(|c_V|^2+|c_A|^2)\underbrace{\left[(p_2\cdot p_3)+2\frac{(p_1\cdot p_2)(p_1\cdot p_3)}{(M_zc)^2}\right]}_{\Diamond}.\end{aligned}$$

In the rest system of the Z,

$$p_1=(M_Zc,\mathbf{0});\quad p_2=\left(\frac{M_Zc}{2},\mathbf{p}\right);\quad p_3=\left(\frac{M_Zc}{2},-\mathbf{p}\right),$$

and (since f is "massless")

$$\left(\frac{M_Zc}{2}\right)^2-\mathbf{p}^2=0;\quad\text{or}\quad \mathbf{p}^2=\left(\frac{M_Zc}{2}\right)^2.$$

So

$$(p_2\cdot p_3)=\left(\frac{M_Zc}{2}\right)^2+\mathbf{p}^2=\frac{(M_Zc)^2}{2};\quad (p_1\cdot p_2)=(p_1\cdot p_3)=\frac{(M_Zc)^2}{2},$$

and hence

$$\Diamond=\frac{(M_Zc)^2}{2}+\not{2}\frac{\frac{(M_Zc)^2}{\not{2}}\frac{(M_Zc)^2}{2}}{(M_Zc)^2}=(M_Zc)^2.$$

Therefore, (restoring now the superscripts on c_V and c_A)

$$\Gamma=\boxed{\frac{g_z^2(M_Zc^2)}{48\pi\hbar}(|c_V^f|^2+|c_A^f|^2)}.$$

(b) Let $A\equiv(|c_V^f|^2+|c_A^f|^2)$. According to Table 9.1:

ν_e,ν_μ,ν_τ :	$c_V=0.5$;	$c_A=0.5$;	$A=0.25+0.25=0.5000$
e,μ,τ :	$c_V=-0.0372$;	$c_A=-0.5$;	$A=0.0014+0.25=0.2514$
u,c,t :	$c_V=0.1915$;	$c_A=0.5$;	$A=0.0367+0.25=0.2867$
d,s,b :	$c_V=-0.3457$;	$c_A=-0.5$;	$A=0.1195+0.25=0.3695.$

Now

$$\Gamma_{\text{total}} = \frac{g_z^2 M_Z c^2}{48\pi\hbar} \sum_f (|c_V^f|^2 + |c_A^f|^2),$$

where the sum is over the three neutrinos, three charged leptons, u and c (but *not* the t, since it is too heavy: $m_t > M_Z/2$), d, s, and b (and in the case of the quarks, three colors):

$$\sum_f (|c_V^f|^2 + |c_A^f|^2) = 3(0.5000) + 3(0.2514) + 6(0.2867) + 9(0.3695) = 7.2999.$$

The branching ratios ($\Gamma_i/\Gamma_{\text{total}}$) are:

$$\begin{aligned}
&e, \mu, \tau &&: \ 0.2514/7.2999 = 0.0344, \text{ or } \boxed{3\%} \\
&\nu_e, \nu_\mu, \nu_\tau &&: \ 0.5000/7.2999 = 0.0685, \text{ or } \boxed{7\%} \\
&u, c &&: \ 3(0.2867)/7.2999 = 0.1178, \text{ or } \boxed{12\%} \\
&d, s, b &&: \ 3(0.3695)/7.2999 = 0.1519, \text{ or } \boxed{15\%}
\end{aligned}$$

(c) From Eq. 9.91,

$$g_z^2 = \frac{g_e^2}{\sin^2\theta_w \cos^2\theta_w} = \frac{4\pi\alpha}{(0.2314)(0.7686)} = 0.5156,$$

so

$$\begin{aligned}
\tau &= \frac{1}{\Gamma_{\text{total}}} = \frac{48\pi\hbar}{g_z^2 M_Z c^2} \frac{1}{(7.2999)} = \frac{48\pi(6.582 \times 10^{-22}\,\text{MeV s}}{(0.5156)(91188\,\text{MeV})(7.2999)} \\
&= \boxed{2.89 \times 10^{-25}\ \text{s}}.
\end{aligned}$$

With a fourth generation, the lifetime would *decrease*. The precise factor depends on how many (and which) members of the 4th generation have masses less than $M_Z/2$, but it should be approximately 3/4.

Problem 9.24

$$\begin{aligned}
R &= \frac{\sigma(e^+e^- \to Z \to \text{hadrons})}{\sigma(e^+e^- \to Z \to \mu^+\mu^-)} = \frac{\sum_q \left[\left(c_V^q\right)^2 + \left(c_A^q\right)^2\right]}{\left[\left(c_V^\mu\right)^2 + \left(c_A^\mu\right)^2\right]} \\
&= \frac{9(0.2867 + 0.3695)}{0.2514} = \boxed{23.5}.
\end{aligned}$$

(The numbers are from Problem 9.23(b).)

Problem 9.25

First evaluate the coefficient in curly brackets:

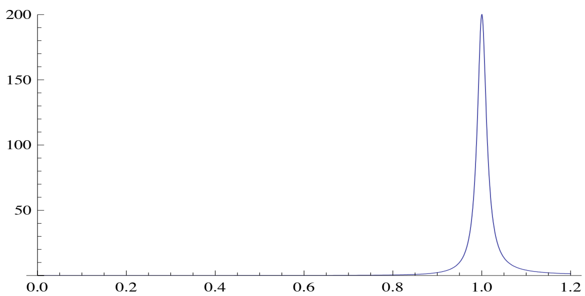
$$\left[\frac{0.5000 - 2(0.2314) + 4(0.2314)^2}{(0.2314)(0.7686)}\right]^2 = 1.998 \approx 2.$$

Then

$$\frac{\sigma_Z}{\sigma_\gamma} = 2\frac{1}{16}\frac{(2E)^4}{[(2E)^2 - (M_Z c^2)^2]^2 + (\hbar\Gamma_Z M_Z c^2)^2} = \frac{1}{8}\frac{x^4}{(x^2-1)^2 + a^2},$$

where (using $g_z^2 = 0.5156$, from Eq. 9.91)

$$a \equiv \frac{\hbar\Gamma_Z}{M_Z c^2} = \frac{7.3 g_z^2}{48\pi} = 0.025.$$



Problem 9.26

(a) Given

$$(\not p - mc)u = 0; \quad u_\pm \equiv \tfrac{1}{2}(1 \pm \gamma^5)u$$

(upper sign for R, lower for L).

$$\begin{aligned}(\not p - mc)u_\pm &= \tfrac{1}{2}(\not p - mc)(1 \pm \gamma^5)u = \cancel{\tfrac{1}{2}(\not p - mc)u} \pm \tfrac{1}{2}(\not p - mc)\gamma^5 u \\ &= \pm\tfrac{1}{2}\gamma^5(-\not p - mc)u = \mp mc\gamma^5 u \neq 0\end{aligned}$$

(unless $m = 0$). I used the fact that γ^5 anticommutes with γ^μ, so $\not p\gamma^5 = -\gamma^5\not p$.

(b)

$$P_\pm = \frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \pm \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}1 & \pm 1\\ \pm 1 & 1\end{pmatrix}.$$

$$P_\pm w = \lambda w \Rightarrow \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}\begin{pmatrix} w_A \\ w_B \end{pmatrix} = \lambda \begin{pmatrix} w_A \\ w_B \end{pmatrix} \Rightarrow \begin{pmatrix} w_A \pm w_B \\ w_B \pm w_A \end{pmatrix} = 2\lambda \begin{pmatrix} w_A \\ w_B \end{pmatrix},$$

so

$$w_A \pm w_B = 2\lambda w_A,\ w_B \pm w_A = 2\lambda w_B;\ (2\lambda - 1)w_A = \pm w_B,\ (2\lambda - 1)w_B = \pm w_A.$$

$$(2\lambda - 1)^2 w_A = \pm(2\lambda - 1)w_B = w_A, \quad (4\lambda^2 - 4\lambda + 1)w_A = w_A,$$

and hence

$$\lambda(\lambda - 1) = 0, \quad \text{or} \quad \boxed{\lambda = 0,\ \lambda = 1}.$$

If $\lambda = 0$, $w_B = \mp w_A$; if $\lambda = 1$, $w_B = \pm w_A$. So a simple set of eigenspinors would be

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad (\lambda = 0 \text{ for } P_+,\ \lambda = 1 \text{ for } P_-);$$

$$w_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad w_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad (\lambda = 1 \text{ for } P_+,\ \lambda = 0 \text{ for } P_-).$$

(c)

$$\begin{aligned}
[P_\pm, (\not{p} - mc)] &= \tfrac{1}{2}\left\{(1 \pm \gamma^5)(\not{p} - mc) - (\not{p} - mc)(1 \pm \gamma^5)\right\} \\
&= \tfrac{1}{2}\left\{(\not{p} - mc) - (\not{p} - mc) \pm \left[\gamma^5(\not{p} - mc) - (\not{p} - mc)\gamma^5\right]\right\} \\
&= \pm\tfrac{1}{2}\gamma^5(\not{p} - mc + \not{p} + mc) = \pm\gamma^5\not{p} \neq 0.
\end{aligned}$$

But if a spinor is simultaneously an eigenstate of two noncommuting operators, A and B, which is to say, if $Aw = \lambda w$ and $Bw = \mu w$, then

$$[A, B]w = ABw - BAw = \mu(Aw) - \lambda(Bw) = \mu\lambda w - \lambda\mu w = 0.$$

So w could be an eigenspinor of both $P_\pm$ and $(\not{p} - mc)$ only if $\gamma^5\not{p}w = 0$, or $m\gamma^5 w = 0$, which is only possible if $m = 0$ (if $\gamma^5 w = 0$, then $P_\pm w = (1/2)w = 0$, and $w = 0$ is by definition not an eigenspinor).

Problem 9.27

$$\chi_L = \begin{pmatrix} u \\ d' \end{pmatrix}_L \ (9.141); \quad j^\pm_\mu = \bar{\chi}_L \gamma_\mu \tau^\pm \chi_L \ (9.136); \quad \tau^\pm = \tfrac{1}{2}(\tau^1 \pm i\tau^2) \ (9.137).$$

$$\tau^+ = \frac{1}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix};$$
$$\tau^- = \frac{1}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

$$\begin{aligned} j^+_\mu &= \left(\bar{u}\ \bar{d}'\right)_L \gamma_\mu \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u\\ d'\end{pmatrix}_L = \left(\bar{u}\ \bar{d}'\right)_L \gamma_\mu \begin{pmatrix}d'\\ 0\end{pmatrix}_L = \bar{u}_L\gamma_\mu d'_L \\ &= \bar{u}\tfrac{1}{2}(1+\gamma^5)\gamma_\mu\tfrac{1}{2}(1-\gamma^5)d' = \tfrac{1}{4}\bar{u}\gamma_\mu(1-\gamma^5)^2 d' = \boxed{\tfrac{1}{2}\bar{u}\gamma_\mu(1-\gamma^5)d'}. \end{aligned}$$

I used Table 9.2 for the chiral spinors, $\{\gamma^5, \gamma_\mu\} = 0$, and $(1-\gamma^5)^2 = 2(1-\gamma^5)$.

$$\begin{aligned} j^-_\mu &= \left(\bar{u}\ \bar{d}'\right)_L \gamma_\mu \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}u\\ d'\end{pmatrix}_L = \left(\bar{u}\ \bar{d}'\right)_L \gamma_\mu \begin{pmatrix}0\\ u\end{pmatrix}_L = \bar{d}'_L\gamma_\mu u_L \\ &= \bar{d}'\tfrac{1}{2}(1+\gamma^5)\gamma_\mu\tfrac{1}{2}(1-\gamma^5)u = \tfrac{1}{4}\bar{d}'\gamma_\mu(1-\gamma^5)^2 u = \boxed{\tfrac{1}{2}\bar{d}'\gamma_\mu(1-\gamma^5)u}. \end{aligned}$$

Meanwhile, from Eq . 9.138,

$$\begin{aligned} j^3_\mu &= \tfrac{1}{2}\bar{\chi}_L\gamma_\mu\tau^3\chi_L = \tfrac{1}{2}\left(\bar{u}\ \bar{d}'\right)_L \gamma_\mu \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}u\\ d'\end{pmatrix}_L = \tfrac{1}{2}\left(\bar{u}\ \bar{d}'\right)_L \gamma_\mu \begin{pmatrix}u\\ -d'\end{pmatrix}_L \\ &= \tfrac{1}{2}\left[\bar{u}_L\gamma_\mu u_L - \bar{d}'_L\gamma_\mu d'_L\right] = \boxed{\tfrac{1}{4}\left[\bar{u}\gamma_\mu(1-\gamma^5)u - \bar{d}'\gamma_\mu(1-\gamma^5)d'\right]}, \end{aligned}$$

and, from Eqs. 9.144 and 9.131,

$$j^{em}_\mu = \sum_{i=1}^{2} Q_i(\bar{u}_i\gamma_\mu u_i) = \boxed{\left[\tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}'\gamma_\mu d'\right]}.$$

Finally, from Eq. 9.140,

$$\begin{aligned} j^Y_\mu &= 2(j^{em}_\mu - j^3_\mu) = \tfrac{4}{3}\bar{u}\gamma_\mu u - \tfrac{2}{3}\bar{d}'\gamma_\mu d' - \tfrac{1}{2}\bar{u}\gamma_\mu(1-\gamma^5)u + \tfrac{1}{2}\bar{d}'\gamma_\mu(1-\gamma^5)d' \\ &= \boxed{\tfrac{1}{2}\left[\bar{u}\gamma_\mu\left(\tfrac{5}{3}+\gamma^5\right)u - \bar{d}'\gamma_\mu\left(\tfrac{1}{3}+\gamma^5\right)d'\right]}. \end{aligned}$$

Problem 9.28

According to Eq. 9.155, the quark and lepton couplings to the Z are of the form

$$-ig_z\left(j^3_\mu - \sin^2\theta_w\, j^{em}_\mu\right).$$

I'll do the first generation (e, ν_e, u, d); the other generations are identical. (As explained in the first footnote to Section 9.6, it doesn't matter whether you use d or d', so I'll use d.) For the quarks (Problem 9.27 and Eq. 9.144):

$$j^3_\mu = \tfrac{1}{4}\left[\bar{u}\gamma_\mu(1-\gamma^5)u - \bar{d}\gamma_\mu(1-\gamma^5)d\right]; \quad j^{em}_\mu = \tfrac{2}{3}(\bar{u}\gamma_\mu u) - \tfrac{1}{3}(\bar{d}\gamma_\mu d).$$

For the leptons (Eq. 9.138):

$$j^3_\mu = \tfrac{1}{2}[\bar{\nu}_L\gamma_\mu\nu_L - \bar{e}_L\gamma_\mu e_L].$$

But $\bar{\nu}_L\gamma_\mu\nu_L = \bar{\nu}\frac{1}{2}(1+\gamma^5)\gamma_\mu\frac{1}{2}(1-\gamma^5)\nu$ (and similarly for the electron term), while $(1+\gamma^5)\gamma_\mu = \gamma_\mu(1-\gamma^5)$, and $(1-\gamma^5)^2 = 2(1-\gamma^5)$. So

$$j^3_\mu = \tfrac{1}{4}[\bar{\nu}\gamma_\mu(1-\gamma^5)\nu - \bar{e}\gamma_\mu(1-\gamma^5)e], \quad \text{while} \quad j^{em}_\mu = -(\bar{e}\gamma_\mu e)$$

(Eq. 9.131).

The generic neutral weak vertex factor is (Eq. 9.90)

$$-\frac{ig_z}{2}\gamma_\mu(c_V - c_A\gamma^5);$$

picking out the relevant terms in $-ig_z\left(j^3_\mu - \sin^2\theta_w\, j^{em}_\mu\right)$, we find:

Neutrino:

$$-ig_z\left[\tfrac{1}{4}\bar{\nu}\gamma_\mu(1-\gamma^5)\nu\right] = -\frac{ig_z}{2}\left[\bar{\nu}\gamma_\mu\left(\tfrac{1}{2} - \tfrac{1}{2}\gamma^5\right)\nu\right] \Rightarrow \boxed{c_V = \tfrac{1}{2},\ c_A = \tfrac{1}{2}}.$$

Electron:

$$-ig_z\left[-\tfrac{1}{4}\bar{e}\gamma_\mu(1-\gamma^5)e + \sin^2\theta_w\bar{e}\gamma_\mu e\right] = -\frac{ig_z}{2}\left[\bar{e}\gamma_\mu\left(-\tfrac{1}{2} + 2\sin^2\theta_w + \tfrac{1}{2}\gamma^5\right)e\right]$$

$$\Rightarrow \boxed{c_V = -\tfrac{1}{2} + 2\sin^2\theta_w,\ c_A = -\tfrac{1}{2}}.$$

Up Quark:

$$-ig_z\left[\tfrac{1}{4}\bar{u}\gamma_\mu(1-\gamma^5)u - \tfrac{2}{3}\sin^2\theta_w\bar{u}\gamma_\mu u\right] = -\frac{ig_z}{2}\left[\bar{u}\gamma_\mu\left(\tfrac{1}{2} - \tfrac{4}{3}\sin^2\theta_w - \tfrac{1}{2}\gamma^5\right)u\right]$$

$$\Rightarrow \boxed{c_V = \tfrac{1}{2} - \tfrac{4}{3}\sin^2\theta_w,\ c_A = \tfrac{1}{2}}.$$

Down Quark:

$$-ig_z\left[-\tfrac{1}{4}\bar{d}\gamma_\mu(1-\gamma^5)d + \tfrac{1}{3}\sin^2\theta_w\bar{d}\gamma_\mu d\right] = -\frac{ig_z}{2}\left[\bar{d}\gamma_\mu\left(-\tfrac{1}{2} + \tfrac{2}{3}\sin^2\theta_w + \tfrac{1}{2}\gamma^5\right)d\right]$$

$$\Rightarrow \boxed{c_V = -\tfrac{1}{2} + \tfrac{2}{3}\sin^2\theta_w,\ c_A = -\tfrac{1}{2}}.$$

Problem 9.29

The two basic hadronic modes,

$$(1)\ \tau \to \nu_\tau + d + \bar{u}, \qquad (2)\ \tau \to \nu_\tau + s + \bar{u},$$

have decay rates

$$\Gamma_1 = 3\cos^2\theta_C \Gamma_e, \qquad \Gamma_2 = 3\sin^2\theta_C \Gamma_e$$

(3 for color), where Γ_e is the rate for $\tau \to \nu_\tau + e + \bar{\nu}_e$ (from Problem 9.5). So

$$\Gamma_{\text{tot}} = \Gamma_e + \Gamma_\mu + \Gamma_1 + \Gamma_2 = 2\Gamma_e + 3\Gamma_e(\sin^2\theta_C + \cos^2\theta_C) = 5\Gamma_e.$$

Evidently

$$\tau = \frac{1}{\Gamma_{\text{tot}}} = \frac{1}{5\Gamma_e} = \frac{2\tau_{\text{leptons}}}{5} = (0.4)(8.16 \times 10^{-13}) = \boxed{3.26 \times 10^{-13}\,\text{s}}$$

(the experimental value lifetime is 2.91×10^{-13} s), and the branching ratios are

$$\frac{\Gamma_e}{\Gamma_{\text{tot}}} = \frac{\Gamma_\mu}{\Gamma_{\text{tot}}} = \frac{1}{5} = \boxed{20\%}, \qquad \frac{\Gamma_{\text{hadrons}}}{\Gamma_{\text{tot}}} = \frac{3}{5} = \boxed{60\%}.$$

The experimental branching ratios are 17.8% (electrons), 17.4% (muons), and 61% for hadrons. [I got the latter figure from the Particle Physics Booklet, by adding the dominant hadronic modes: $\pi^-\pi^0\nu_\tau$ (25.5%), $\pi^-\nu_\tau$ (10.9%), $\pi^-\pi^-\pi^+\nu_\tau$ (9.3%), $\pi^-\pi^0\pi^0\nu_\tau$ (9.3%), $\pi^-\pi^-\pi^+\pi^0\nu_\tau$ (4.6%), $\pi^-\pi^0\pi^0\pi^0\nu_\tau$ (1.0%). This doesn't count the many modes with branching ratios below 1%; by my estimate they add to about 4%, raising the total hadronic ratio to about 65%.]

Problem 9.30

(a) Treat all secondaries as massless. The principal decay modes are

$$
\begin{aligned}
&c \to s + e^+ + \nu_e: && \Gamma = |V_{cs}|^2\Gamma_0 \\
&c \to s + \mu^+ + \nu_\mu: && \Gamma = |V_{cs}|^2\Gamma_0 \\
&c \to d + e^+ + \nu_e: && \Gamma = |V_{cd}|^2\Gamma_0 \\
&c \to d + \mu^+ + \nu_\mu: && \Gamma = |V_{cd}|^2\Gamma_0 \\
&c \to s + u + \bar{d}: && \Gamma = 3|V_{cs}|^2|V_{ud}|^2\Gamma_0 \\
&c \to s + u + \bar{s}: && \Gamma = 3|V_{cs}|^2|V_{us}|^2\Gamma_0 \\
&c \to d + u + \bar{d}: && \Gamma = 3|V_{cd}|^2|V_{ud}|^2\Gamma_0 \\
&c \to d + u + \bar{s}: && \Gamma = 3|V_{cd}|^2|V_{us}|^2\Gamma_0
\end{aligned}
$$

where

$$\Gamma_0 \equiv \left(\frac{m_c}{m_\mu}\right)^5 \Gamma_\mu,$$

and Γ_μ is the decay rate of the muon. In terms of the Cabibbo angle(stick with CKM matrix elements if you prefer),

$$
\begin{aligned}
\Gamma_{\text{tot}} &= \Gamma_0(\cos^2\theta_C + \cos^2\theta_C + \sin^2\theta_C + \sin^2\theta_C + 3\cos^2\theta_C\cos^2\theta_C \\
&\quad + 3\cos^2\theta_C\sin^2\theta_C + 3\sin^2\theta_C\cos^2\theta_C + 3\sin^2\theta_C\sin^2\theta_C) \\
&= \Gamma_0\left(\cos^2\theta_C + \sin^2\theta_C\right)\left[2 + 3\left(\cos^2\theta_C + \sin^2\theta_C\right)\right] = 5\Gamma_0.
\end{aligned}
$$

So the lifetime is

$$\tau = \frac{1}{5}\left(\frac{m_\mu}{m_c}\right)^5 \tau_\mu = \frac{1}{5}\left(\frac{105.66}{1250}\right)^5 (2.197 \times 10^{-6}) = \boxed{1.9 \times 10^{-12}\,\text{s}}.$$

(b) In this (pretty gross) approximation, the lifetime of the D should be about $\boxed{1.9 \times 10^{-12}\,\text{s}}$. The observed lifetime of the D^+ is 1.04×10^{-12} s, and that of D^0 is 0.41×10^{-12} s, so we're not *too* far off.

The branching ratios for semileptonic (e or μ) and hadronic modes are the same as for the τ (Problem 9.29): 20% (each) and 60%, respectively. According to the Particle Physics Booklet the dominant electronic decays of the D^+ are $\bar{K}^0 e^+ \nu_e$ (8.6%), $\bar{K}^{*0} e^+ \nu_e$ (5.6%), $K^- \pi^+ e^+ \nu_e$ (4.5%), for a total of 18.7%, and the dominant muonic decays are $\bar{K}^0 \mu^+ \nu_\mu$ (9.5%), $\bar{K}^{*0} \mu^+ \nu_\mu$ (5.5%), $K^- \pi^+ \mu^+ \nu_\mu$ (4.0%), for a total of 19.0%. The rest are hadronic, so the agreement is very good. The D^0 decays are much more complicated, and the agreement is not so hot: a total of 6.2% for the electronic modes and 5.6% for muonic modes.

(c) The principal decay modes of the b quark are

$$\begin{aligned}
b &\to c + e + \bar{\nu}_e : & \Gamma &= |V_{cb}|^2 \Gamma_0 \\
b &\to c + \mu + \bar{\nu}_\mu : & \Gamma &= |V_{cb}|^2 \Gamma_0 \\
b &\to c + \tau + \bar{\nu}_\tau : & \Gamma &= |V_{cb}|^2 \Gamma_0 \\
b &\to u + e + \bar{\nu}_e : & \Gamma &= |V_{ub}|^2 \Gamma_0 \\
b &\to u + \mu + \bar{\nu}_\mu : & \Gamma &= |V_{ub}|^2 \Gamma_0 \\
b &\to u + \tau + \bar{\nu}_\tau : & \Gamma &= |V_{ub}|^2 \Gamma_0 \\
b &\to c + d + \bar{u} : & \Gamma &= 3|V_{cb}|^2 |V_{ud}|^2 \Gamma_0 \\
b &\to c + s + \bar{u} : & \Gamma &= 3|V_{cb}|^2 |V_{us}|^2 \Gamma_0 \\
b &\to c + d + \bar{c} : & \Gamma &= 3|V_{cb}|^2 |V_{cd}|^2 \Gamma_0 \\
b &\to c + s + \bar{c} : & \Gamma &= 3|V_{cb}|^2 |V_{cs}|^2 \Gamma_0 \\
b &\to u + d + \bar{u} : & \Gamma &= 3|V_{ub}|^2 |V_{ud}|^2 \Gamma_0 \\
b &\to u + s + \bar{u} : & \Gamma &= 3|V_{ub}|^2 |V_{us}|^2 \Gamma_0 \\
b &\to u + d + \bar{c} : & \Gamma &= 3|V_{ub}|^2 |V_{cd}|^2 \Gamma_0 \\
b &\to u + s + \bar{c} : & \Gamma &= 3|V_{ub}|^2 |V_{cs}|^2 \Gamma_0
\end{aligned}$$

where

$$\Gamma_0 \equiv \left(\frac{m_b}{m_\mu} \right)^5 \Gamma_\mu .$$

So

$$\begin{aligned}
\Gamma_{\text{tot}} &= \Gamma_0 \left[3|V_{cb}|^2 + 3|V_{ub}|^2 + 3(|V_{cb}|^2 + |V_{ub}|^2) \left(|V_{ud}|^2 + |V_{us}|^2 + |V_{cd}|^2 + |V_{cs}|^2 \right) \right] \\
&= 3\Gamma_0 \left(|V_{cb}|^2 + |V_{ub}|^2 \right) \left[1 + \left(\cos^2\theta_C + \sin^2\theta_C + \sin^2\theta_C + \cos^2\theta_C \right) \right] \\
&= 9 \left((|V_{cb}|^2 + |V_{ub}|^2 \right) \Gamma_0 = 9 \left[(0.0422)^2 + (0.0040)^2 \right] \Gamma_0 = 0.0162\, \Gamma_0 .
\end{aligned}$$

So the lifetime of the b quark (and hence of the B meson) is

$$\tau = \frac{1}{0.0162} \left(\frac{m_\mu}{m_b} \right)^5 \tau_\mu = \frac{1}{0.0162} \left(\frac{105.66}{4300} \right)^5 (2.197 \times 10^{-6}) = \boxed{1.2 \times 10^{-12}\,\text{s}} .$$

The observed lifetimes are: 1.64×10^{-12} s (for the B^-), 1.53×10^{-12} s (for the B^0). The agreement is very good for both of them.

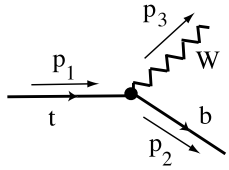
Branching ratios:

$$\frac{\Gamma_\ell}{\Gamma_{\text{tot}}} = \frac{\Gamma_0 \left(|V_{cb}|^2 + |V_{ub}|^2 \right)}{9\Gamma_0 \left(|V_{cb}|^2 + |V_{ub}|^2 \right)} = \frac{1}{9} = \boxed{11\%} ; \quad \frac{\Gamma_{\text{had}}}{\Gamma_{\text{tot}}} = \frac{2}{3} = \boxed{67\%} .$$

The experimental leptonic branching ratios are 10.2% (for the B^+) and 10.5% (for the B^0)—not bad!

(d) The b quark cannot decay without crossing generations, whereas the c quark can go to an s, in the same generation. The smallness of the CKM matrix elements V_{cb} and V_{ub} almost exactly compensates for the larger mass of the b quark, so the final lifetimes come out practically the same.

Problem 9.31



$$\begin{aligned}
\mathcal{M} &= i\bar{u}(2)\left[\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\right]u(1)\epsilon^*_\mu(3)\\
|\mathcal{M}|^2 &= \frac{g_w^2}{8}[\bar{u}(2)\gamma^\mu(1-\gamma^5)u(1)][\bar{u}(1)\gamma^\nu(1-\gamma^5)u(2)]\epsilon^*_\mu(3)\epsilon_\nu(3)\\
\langle|\mathcal{M}|^2\rangle &= \frac{g_w^2}{16}\mathrm{Tr}[\gamma^\mu(1-\gamma^5)(\not{p}_1+m_1c)\gamma^\nu(1-\gamma^5)(\not{p}_2+m_2c)]\left[-g_{\mu\nu}+\frac{p_{3\,\mu}p_{3\,\nu}}{(M_Wc)^2}\right]
\end{aligned}$$

The trace reduces to

$$\begin{aligned}
&\mathrm{Tr}[\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)\not{p}_2]+m_1m_2c^2\mathrm{Tr}[\gamma^\mu(1-\gamma^5)\gamma^\nu(1-\gamma^5)]\\
&= \mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^\nu(1-\gamma^5)^2\not{p}_2]+m_1m_2c^2\mathrm{Tr}[\gamma^\mu\gamma^\nu\underbrace{(1+\gamma^5)(1-\gamma^5)}_{1-(\gamma^5)^2=0}]\\
&= 2\mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^\nu(1-\gamma^5)\not{p}_2] = 2[\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2)+\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2\gamma^5)]\\
&= 2p_{1\kappa}p_{2\lambda}[\mathrm{Tr}(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda)+\mathrm{Tr}(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda\gamma^5)]\\
&= 8p_{1\kappa}p_{2\lambda}\left(g^{\mu\kappa}g^{\nu\lambda}-g^{\mu\nu}g^{\kappa\lambda}+g^{\mu\lambda}g^{\nu\kappa}+i\epsilon^{\mu\kappa\nu\lambda}\right).
\end{aligned}$$

Because this is contracted with a symmetric tensor, the antisymmetric (final) term does not contribute.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{g_w^2}{2}\left[p_1^\mu p_2^\nu-g^{\mu\nu}(p_1\cdot p_2)+p_1^\nu p_2^\mu\right]\left[-g_{\mu\nu}+\frac{p_{3\,\mu}p_{3\,\nu}}{(M_Wc)^2}\right]\\
&= \frac{g_w^2}{2}\left[2(p_1\cdot p_2)+\frac{2(p_1\cdot p_3)(p_2\cdot p_3)-(p_1\cdot p_2)p_3^2}{(M_Wc)^2}\right]\\
&= \frac{g_w^2}{2}\left[(p_1\cdot p_2)+2\frac{(p_1\cdot p_3)(p_2\cdot p_3)}{(M_Wc)^2}\right].
\end{aligned}$$

In the rest frame of the t,

$$p_1 = (m_t c, \mathbf{0}), \quad p_2 = \left(\frac{E_b}{c}, \mathbf{p}\right), \quad p_3 = \left(\frac{E_W}{c}, -\mathbf{p}\right),$$

So

$$p_1 \cdot p_2 = m_t E_b, \quad p_1 \cdot p_3 = m_t E_W, \quad p_2 \cdot p_3 = \frac{E_b E_W}{c^2} + \mathbf{p}^2;$$

$$E_W^2 - \mathbf{p}^2 c^2 = M_W^2 c^4 \Rightarrow \mathbf{p}^2 = \frac{E_W^2}{c^2} - M_W^2 c^2 \Rightarrow p_2 \cdot p_3 = \frac{E_b E_W}{c^2} + \frac{E_W^2}{c^2} - M_W^2 c^2;$$

$$m_t c^2 = E_b + E_W \Rightarrow E_b = m_t c^2 - E_W \Rightarrow p_2 \cdot p_3 = m_t E_W - M_W^2 c^2;$$

$$p_1 \cdot p_2 = m_t(m_t c^2 - E_W) = m_t^2 c^2 - m_t E_W.$$

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{g_w^2}{2}\left[m_t^2 - m_t E_W + \frac{2}{(M_W c)^2} m_t E_W (m_t E_W - M_W^2 c^2)\right] \\
&= \frac{g_w^2}{2}\left[m_t^2 c^2 - m_t E_W - 2 m_t E_W + 2\frac{(m_t E_W)^2}{(M_W c)^2}\right] \\
&= \frac{g_w^2 m_t}{2}\left[m_t c^2 - 3E_W + 2\frac{m_t E_W^2}{(M_W c)^2}\right].
\end{aligned}$$

Using the result of Problem 3.19 (and setting $m_b \to 0$):

$$E_W = \frac{m_t^2 + M_W^2 - m_b^2}{2m_t} c^2 \approx \frac{m_t^2 + M_W^2}{2m_t} c^2.$$

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{g_w^2 m_t}{2}\left[m_t c^2 - \frac{3}{2}\frac{(m_t^2 + M_W^2)c^2}{m_t} + \frac{2m_t}{(M_W c)^2}\frac{(m_t^4 + 2m_t^2 M_W^2 + M_W^4)c^4}{4m_t^2}\right] \\
&= \frac{g_w^2}{2}\left(m_t^2 c^2 - \frac{3}{2} m_t^2 c^2 - \frac{3}{2} M_W^2 c^2 + \frac{1}{2}\frac{m_t^4 c^4}{M_W^2 c^2} + m_t^2 c^2 + \frac{1}{2} M_W^2 c^2\right) \\
&= \frac{g_w^2 c^2}{4M_W^2}\left(m_t^2 M_W^2 - 2M_W^4 + m_t^4\right) = \left(\frac{g_w c}{2M_W}\right)^2 (m_t^2 - M_W^2)(m_t^2 + 2M_W^2).
\end{aligned}$$

Quoting Eq. 6.35,

$$\Gamma = \frac{|\mathbf{p}|}{8\pi\hbar m_t^2 c}\left(\frac{g_w c}{2M_W}\right)^2 (m_t^2 - M_W^2)(m_t^2 + 2M_W^2).$$

But (Problem 3.19b, with $m_b \to 0$)

$$|\mathbf{p}| \approx \frac{\sqrt{m_t^4 + M_W^4 - 2m_t^2 M_W^2}}{2m_t} c = \frac{(m_t^2 - M_W^2)c}{2m_t},$$

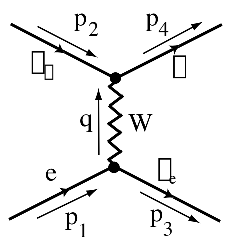
so

$$\begin{aligned}\Gamma &= \frac{g_w^2 (m_t c^2)^3}{64\pi\hbar (M_W c^2)^2}\left[1-\left(\frac{M_W}{m_t}\right)^2\right]^2\left[1+2\left(\frac{M_W}{m_t}\right)^2\right] \\ &= \frac{(0.653)^2(174000)^3}{64\pi(6.582\times 10^{-22})(80403)^2}\left[1-\left(\frac{80.4}{174}\right)^2\right]^2\left[1+2\left(\frac{80.4}{174}\right)^2\right] \\ &= 2.32\times 10^{24}\,/\mathrm{s}.\end{aligned}$$

$$\tau = 1/\Gamma = \boxed{4.3\times 10^{-25}\,\mathrm{s}}.$$

Problem 9.32

(a)



$$\int \left\{\bar{u}(3)\left[\frac{-ig_w}{2\sqrt{2}}(1-\gamma^5)\right]u(1)\right\}\frac{i}{(M_W c)^2}\left\{\bar{u}(4)\left[\frac{-ig_w}{2\sqrt{2}}(1-\gamma^5)\right]u(2)\right\}$$

$$\times(2\pi)^4\delta^4(p_1-p_3-q)(2\pi)^4\delta^4(p_2-p_4+q)\frac{d^4q}{(2\pi)^4}.$$

Using the first delta function, the integral replaces q by p_1-p_3; erasing $(2\pi)^4\delta(p_1+p_2-p_3-p_4)$ and multiplying by i leaves

$$\mathcal{M} = \boxed{\frac{g_w^2}{8(M_W c)^2}\left[\bar{u}(3)(1-\gamma^5)u(1)\right]\left[\bar{u}(4)(1-\gamma^5)u(2)\right]}.$$

(b)

$$\begin{aligned}|\mathcal{M}|^2 &= \left[\frac{g_w^2}{8(M_W c)^2}\right]^2\left[\bar{u}(3)(1-\gamma^5)u(1)\right]\left[\bar{u}(3)(1-\gamma^5)u(1)\right]^* \\ &\qquad\times\left[\bar{u}(4)(1-\gamma^5)u(2)\right]\left[\bar{u}(4)(1-\gamma^5)u(2)\right]^*.\end{aligned}$$

$$\left[\bar{u}(3)(1-\gamma^5)u(1)\right]^* = \left[u(3)^\dagger\gamma^0(1-\gamma^5)u(1)\right]^\dagger = u(1)^\dagger(1-\gamma^5)^\dagger(\gamma^0)^\dagger u(3)$$
$$= \bar{u}(1)\gamma^0(1-\gamma^5)\gamma^0 u(3) = \bar{u}(1)(1+\gamma^5)\gamma^0\gamma^0 u(3)$$
$$= \bar{u}(1)(1+\gamma^5)u(3).$$

$$|\mathcal{M}|^2 = \left[\frac{g_w^2}{8(M_W c)^2}\right]^2 \left[\bar{u}(3)(1-\gamma^5)u(1)\right]\left[\bar{u}(1)(1+\gamma^5)u(3)\right]$$
$$\times \left[\bar{u}(4)(1-\gamma^5)u(2)\right]\left[\bar{u}(2)(1+\gamma^5)u(4)\right].$$

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\left[\frac{g_w^2}{8(M_W c)^2}\right]^2 \mathrm{Tr}\left[(1-\gamma^5)(\not{p}_1 + m_1 c)(1+\gamma^5)(\not{p}_3 + m_3 c)\right]$$
$$\times \mathrm{Tr}\left[(1-\gamma^5)(\not{p}_2 + m_2 c)(1+\gamma^5)(\not{p}_4 + m_4 c)\right].$$

But $m_2 = m_3 = 0$, so the $m_1 c$ in the first term (and $m_4 c$ in the last) are traces of an odd number of gamma matrices, and hence are zero.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\left[\frac{g_w^2}{8(M_W c)^2}\right]^2 \mathrm{Tr}\left[(1-\gamma^5)\not{p}_1(1+\gamma^5)\not{p}_3\right]\mathrm{Tr}\left[(1-\gamma^5)\not{p}_2(1+\gamma^5)\not{p}_4\right]$$

But $\not{p}(1+\gamma^5) = (1-\gamma^5)\not{p}$, and $(1-\gamma^5)^2 = 2(1-\gamma^5)$, so

$$\langle|\mathcal{M}|^2\rangle = 2\left[\frac{g_w^2}{8(M_W c)^2}\right]^2 \mathrm{Tr}\left[(1-\gamma^5)\not{p}_1\not{p}_3\right]\mathrm{Tr}\left[(1-\gamma^5)\not{p}_2\not{p}_4\right].$$

Now, $\mathrm{Tr}(\gamma^5\not{a}\not{b}) = 0$, and $\mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b$, so

$$\langle|\mathcal{M}|^2\rangle = \boxed{\frac{1}{2}\left[\frac{g_w^2}{(M_W c)^2}\right]^2 (p_1\cdot p_3)(p_2\cdot p_4)}.$$

(c) Equation 6.47 says

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\langle|\mathcal{M}|^2\rangle}{(E_1+E_2)^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}.$$

Treating all four particles as massless,

$$p_1 = \left(\frac{E}{c}, \mathbf{p}_i\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}_i\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}_f\right), \quad p_4 = \left(\frac{E}{c}, -\mathbf{p}_f\right);$$

$$|\mathbf{p}_i| = |\mathbf{p}_f| = \frac{E}{c}.$$

So

$$p_1 \cdot p_3 = \left(\frac{E}{c}\right)^2 - \mathbf{p}_i \cdot \mathbf{p}_f = \left(\frac{E}{c}\right)^2 - \left(\frac{E}{c}\right)^2 \cos\theta = 2\left(\frac{E}{c}\right)^2 \sin^2(\theta/2),$$
$$p_2 \cdot p_4 = \left(\frac{E}{c}\right)^2 - \mathbf{p}_i \cdot \mathbf{p}_f = \left(\frac{E}{c}\right)^2 - \left(\frac{E}{c}\right)^2 \cos\theta = 2\left(\frac{E}{c}\right)^2 \sin^2(\theta/2)$$

(where θ is the angle between the incoming electron and the outgoing neutrino).

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{4E^2}\frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4 \left[\left(\frac{E}{c}\right)^2 2\sin^2(\theta/2)\right]^2$$
$$= \boxed{\frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4 \left(\frac{\hbar}{8\pi c}\right)^2 E^2 \sin^4(\theta/2)}.$$

(d)

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4 \left(\frac{\hbar}{8\pi c}\right)^2 E^2 \int \frac{1}{4}(1-\cos\theta)^2 \sin\theta \, d\theta \, d\phi$$
$$= \frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4 \left(\frac{\hbar}{8\pi c}\right)^2 E^2 \frac{2\pi}{4} \underbrace{\int_0^\pi (1-\cos\theta)^2 \sin\theta \, d\theta}_{\frac{1}{3}(1-\cos\theta)^3\big|_0^\pi = \frac{8}{3}}$$
$$= \frac{2}{3\pi}\left(\frac{g_w}{M_W c}\right)^4 \left(\frac{\hbar}{8c}\right)^2 E^2 = \boxed{\frac{2}{3\pi}\left(\frac{\hbar g_w^2 E}{8 M_W^2 c^3}\right)^2}.$$

(e) The Standard Model (Eqs. 9.13 and 9.14, with $m_\mu \to 0$) says

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{\hbar c g_w^2 E}{4\pi (M_W c)^2}\right)^2, \qquad \sigma = \frac{1}{8\pi}\left(\frac{\hbar g_w^2 E}{M_W^2 c^3}\right)^2.$$

Both the Griffiths theory and the Standard Model say $d\sigma/d\Omega$ and σ are proportional to E^2, so the energy dependence offers no test. The obvious difference is that the Standard Model says $d\sigma/d\Omega$ is independent of the scattering angle, whereas my theory says it goes like $\sin^4(\theta/2)$—it's *zero* in the forward direction ($\theta = 0$), and rises to a maximum at $\theta = 180°$ (back-scattering). So the thing to do is $\boxed{\text{count the number of muons coming off at various directions}}$; if it's constant, the Standard Model wins, but I say more of them will come out along the direction of the incident electron.

Problem 9.33

From the 2006 Particle Physics Booklet, Section 11.2:

$$|V_{ud}| = 0.97377, \qquad |V_{ub}| = 0.00431, \qquad |V_{cb}| = 0.230,$$
$$|V_{cb}| = 0.0416, \qquad |V_{td}| = 0.0074, \qquad |V_{ub}| = 0.9991,$$

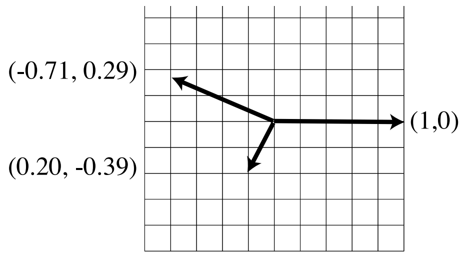
so

$$|z_1| = \left|\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right| = \frac{(0.97377)(0.00431)}{(0.230)(0.0416)} = 0.44;$$
$$|z_2| = \left|\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right| = \frac{(0.0074)(0.9991)}{(0.230)(0.0416)} = 0.77.$$

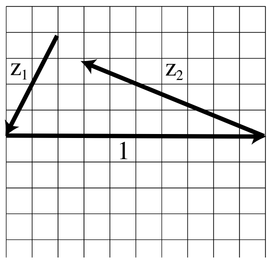
The phases are discussed in the Particle Physics Booklet, Section 11.3:

$$\gamma \equiv \arg(-z_1) = 63^\circ; \quad \beta \equiv \arg(-1/z_2) = \tfrac{1}{2}\sin^{-1}(0.687) = 22^\circ.$$

If we write $z_2 = |z_2|e^{i\theta}$, then $-1/z_2 = (1/|z_2|)e^{i(\pi-\theta)}$, so $(\pi - \theta) = \beta$, and hence $\theta = \pi - \beta = 180^\circ - 22^\circ = 158^\circ$. In the complex plane, then, the three numbers 1, z_1, and z_2 look like this:



The unitarity triangle itself takes the form



Here the coordinates of the tail of z_1 are $0.44\cos 63^\circ = 0.20, 0.44\sin 63^\circ = 0.39$; the coordinates of the head of z_2 are $1 - 0.77\cos 22^\circ = 0.29, 0.77\sin 22^\circ = 0.29$. Evidently the triangle does *not* (quite) close, with the current experimental values.

10
Gauge Theories

Problem 10.1

(a) If r is the distance from the apex, then

$$z = r\cos\alpha, \quad \text{so} \quad v_r = \frac{\dot{z}}{\cos\alpha}.$$

The tangential velocity is

$$v_\phi = r\sin\alpha\,\dot{\phi} = z\tan\alpha\,\dot{\phi},$$

so

$$T = \frac{1}{2}m(v_r^2 + v_\phi^2) = \boxed{\frac{m}{2\cos^2\alpha}\left(z^2\sin^2\alpha\,\dot{\phi}^2 + \dot{z}^2\right)}. \quad U = \boxed{mgz}.$$

(b)

$$L = T - U = \boxed{\frac{m}{2\cos^2\alpha}\left(z^2\sin^2\alpha\,\dot{\phi}^2 + \dot{z}^2\right) - mgz}.$$

$$\frac{\partial L}{\partial \dot{z}} = \frac{m}{\cos^2\alpha}\dot{z} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{m}{\cos^2\alpha}\ddot{z}; \quad \frac{\partial L}{\partial z} = \frac{m}{\cos^2\alpha}z\sin^2\alpha\,\dot{\phi}^2 - mg.$$

Euler-Lagrange for z says

$$\frac{m}{\cos^2\alpha}\ddot{z} = \frac{m}{\cos^2\alpha}z\sin^2\alpha\,\dot{\phi}^2 - mg, \text{ or } \boxed{\ddot{z} = z\sin^2\alpha\dot{\phi}^2 - g\cos^2\alpha}.$$

Similarly,

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{m}{\cos^2\alpha}z^2\sin^2\alpha\dot{\phi} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = m\tan^2\alpha\left(2z\dot{z}\dot{\phi} + z^2\ddot{\phi}\right); \quad \frac{\partial L}{\partial \phi} = 0.$$

Euler-Lagrange for ϕ says

$$m\tan^2\alpha\left(2z\dot{z}\dot{\phi} + z^2\ddot{\phi}\right) = 0, \text{ or } \boxed{\ddot{\phi} = -\frac{2\dot{z}\dot{\phi}}{z}}.$$

(c) Since $\partial L/\partial\phi = 0$, the Euler-Lagrange equation for ϕ says $\partial L/\partial\dot{\phi}$ is a constant—call it ℓ: $\ell \equiv mz^2\tan^2\alpha\,\dot{\phi}$. Physically, it is the

$$\boxed{z\text{ component of angular momentum}}.$$

(d)

$$\dot{\phi} = \frac{\ell}{m\tan^2\alpha\, z^2} \Rightarrow \ddot{z} = z\sin^2\alpha\frac{\ell^2}{m^2\tan^4\alpha\, z^4} - g\cos^2\alpha,$$

or

$$\boxed{\ddot{z} = \cos^2\alpha\left(\frac{\ell^2\cot^2\alpha}{m^2z^3} - g\right)}.$$

[Conservation of energy says

$$E = T + U = \frac{m}{2\cos^2\alpha}\left(z^2\sin^2\alpha\,\dot{\phi}^2 + \dot{z}^2\right) + mgz.$$

Using ℓ to eliminate $\dot{\phi}$ now yields an equation for $\dot{z}$.]

Problem 10.2

It is safest to work out a specific component—say $\mu = 0$, $\nu = 1$. First find all the terms in $\mathscr{L}$ that involve $\partial_0 A_1$ (or $\partial^0 A^1$—which, of course, is $-\partial_0 A_1$):

$$\begin{aligned}
&-\frac{1}{16\pi}\left[\left(\partial^0A^1 - \partial^1A^0\right)(\partial_0A_1 - \partial_1A_0) + \left(\partial^1A^0 - \partial^0A^1\right)(\partial_1A_0 - \partial_0A_1) + \cdots\right] \\
&= \frac{-1}{16\pi}\Big[\left(\partial^0A^1\right)(\partial_0A_1) - \left(\partial^0A^1\right)(\partial_1A_0) - \left(\partial^1A^0\right)(\partial_0A_1) + \left(\partial^1A^0\right)(\partial_1A_0) \\
&+ \left(\partial^1A^0\right)(\partial_1A_0) - \left(\partial^1A^0\right)(\partial_0A_1) - \left(\partial^0A^1\right)(\partial_1A_0) + \left(\partial^0A^1\right)(\partial_0A_1)\Big] + \cdots \\
&= \frac{-1}{16\pi}\left[-2\left(\partial_0A_1\right)^2 + 4\left(\partial_0A_1\right)\left(\partial_1A_0\right)\right] + \cdots
\end{aligned}$$

Evidently

$$\begin{aligned}
\frac{\partial\mathscr{L}}{\partial(\partial_0A_1)} &= \frac{-1}{16\pi}\left[-4(\partial_0A_1) + 4(\partial_1A_0)\right] = \frac{-1}{4\pi}\left(-\partial_0A_1 + \partial_1A_0\right) \\
&= \frac{-1}{4\pi}\left(\partial^0A^1 + \partial^1A^0\right).
\end{aligned}$$

The other components are the same (except for a few signs when μ and ν are both spatial).

Problem 10.3

First note that $\partial_\nu\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu\partial^\mu(\partial_\nu A^\nu) - \partial_\nu\partial^\nu(\partial_\mu A^\mu) = 0$. Now apply ∂_ν to Eq. 10.19:

$$\partial_\nu\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \left(\frac{mc}{\hbar}\right)^2 \partial_\nu A^\nu = 0 \Rightarrow \partial_\nu A^\nu = 0. \checkmark$$

Hence the middle term in Eq. 10.19 is actually *zero*, leaving

$$\partial_\mu\partial^\mu(A^\nu) + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0. \checkmark$$

Problem 10.4

$$\frac{\partial\mathscr{L}}{\partial(\partial_\mu\bar{\psi})} = -\frac{i\hbar c}{2}\gamma^\mu\psi \Rightarrow \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\bar{\psi})}\right) = -\frac{i\hbar c}{2}\gamma^\mu\partial_\mu\psi;$$
$$\frac{\partial\mathscr{L}}{\partial\bar{\psi}} = \frac{i\hbar c}{2}\gamma^\mu(\partial_\mu\psi) - (mc^2)\psi.$$

The Euler-Lagrange equation for $\bar{\psi}$ says:

$$-\frac{i\hbar c}{2}\gamma^\mu\partial_\mu\psi = \frac{i\hbar c}{2}\gamma^\mu(\partial_\mu\psi) - (mc^2)\psi, \quad \text{or} \quad i\gamma^\mu(\partial_\mu\psi) - \left(\frac{mc}{\hbar}\right)\psi = 0$$

(the Dirac equation). Likewise

$$\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)} = \frac{i\hbar c}{2}\bar{\psi}\gamma^\mu \Rightarrow \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\right) = \frac{i\hbar c}{2}(\partial_\mu\psi)\gamma^\mu;$$
$$\frac{\partial\mathscr{L}}{\partial\psi} = -\frac{i\hbar c}{2}(\partial_\mu\bar{\psi})\gamma^\mu - (mc^2)\bar{\psi}.$$

The Euler-Lagrange equation for ψ says:

$$\frac{i\hbar c}{2}(\partial_\mu\bar{\psi})\gamma^\mu = -\frac{i\hbar c}{2}(\partial_\mu\bar{\psi})\gamma^\mu - (mc^2)\bar{\psi}, \quad \text{or} \quad i(\partial_\mu\bar{\psi})\gamma^\mu + \left(\frac{mc}{\hbar}\right)\bar{\psi} = 0$$

(the adjoint Dirac equation).

Problem 10.5

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \frac{1}{2}\partial^\mu\phi; \quad \frac{\partial\mathcal{L}}{\partial(\phi^*)} = -\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi.$$

So the Euler–Lagrange equation leads to

$$\partial_\mu\left(\frac{1}{2}\partial^\mu\phi\right) = -\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi, \quad \text{or} \quad \boxed{\partial_\mu\partial^\mu\phi + \left(\frac{mc}{\hbar}\right)^2\phi = 0}.$$

Similarly,

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}\partial^\mu\phi^*; \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^*, \text{ so } \boxed{\partial_\mu\partial^\mu\phi^* + \left(\frac{mc}{\hbar}\right)^2\phi^* = 0}.$$

Thus ϕ and ϕ^* both satisfy the Klein–Gordon equation. Obviously, the second equation is the complex conjugate of the first.

Problem 10.6

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0; \quad \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = i\hbar c\gamma^\mu\partial_\mu\psi - mc^2\psi - q\gamma^\mu\psi A_\mu.$$

So

$$0 = i\hbar c\gamma^\mu\partial_\mu\psi - mc^2\psi - q\gamma^\mu\psi A_\mu, \text{ or } \boxed{i\hbar\gamma^\mu(\partial_\mu\psi) - mc\psi = \frac{q}{c}A_\mu\gamma^\mu\psi}.$$

Similarly,

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\hbar c\bar{\psi}\gamma^\mu; \quad \frac{\partial\mathcal{L}}{\partial\psi} = -mc^2\bar{\psi} - q\bar{\psi}\gamma^\mu A_\mu;$$

$$i\hbar c(\partial_\mu\bar{\psi})\gamma^\mu = -mc^2\bar{\psi} - q\bar{\psi}\gamma^\mu A_\mu, \text{ or } \boxed{i\hbar(\partial_\mu\bar{\psi})\gamma^\mu + mc\bar{\psi} = -\frac{q}{c}\bar{\psi}\gamma^\mu A_\mu}.$$

Problem 10.7

$$\partial_\mu J^\mu = \partial_\mu[cq\bar{\psi}\gamma^\mu\psi] = cq\left[(\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)\right].$$

With electromagnetic coupling the Dirac equation says (Problem 10.6)

$$\gamma^\mu(\partial_\mu\psi) = -i\left[\left(\frac{mc}{\hbar}\right)\psi + \left(\frac{q}{\hbar c}\right)A_\mu\gamma^\mu\psi\right]; \ (\partial_\mu\bar{\psi})\gamma^\mu = i\left[\left(\frac{mc}{\hbar}\right)\bar{\psi} + \left(\frac{q}{\hbar c}\right)\bar{\psi}\gamma^\mu A_\mu\right].$$

Therefore,

$$\partial_\mu J^\mu = cqi\left[\left(\frac{mc}{\hbar}\right)\bar{\psi}\psi + \left(\frac{q}{\hbar c}\right)(\bar{\psi}\gamma^\mu\psi)A_\mu - \left(\frac{mc}{\hbar}\right)\bar{\psi}\psi - \left(\frac{q}{\hbar c}\right)(\bar{\psi}\gamma^\mu\psi)A_\mu\right] = 0. \ \checkmark$$

Problem 10.8

We simply replace the "ordinary" derivatives by "covariant" derivatives (Eq. 10.38), and add on a free Lagrangian for A^μ:

$$\mathcal{L} = \frac{1}{2}\underbrace{(\mathcal{D}_\mu\phi)^*(\mathcal{D}^\mu\phi)}_{\bigstar} - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^*\phi - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}.$$

Here

$$\begin{aligned}\bigstar &= \left(\partial_\mu\phi + i\frac{q}{\hbar c}A_\mu\phi\right)^*\left(\partial^\mu\phi + i\frac{q}{\hbar c}A^\mu\phi\right)\\ &= (\partial_\mu\phi)^*(\partial^\mu\phi) + \frac{iq}{\hbar c}\left[(\partial_\mu\phi)^*A^\mu\phi - A_\mu\phi^*(\partial^\mu\phi)\right] + \left(\frac{q}{\hbar c}\right)^2 A_\mu A^\mu\phi^*\phi.\end{aligned}$$

So

$$\begin{aligned}\mathcal{L} = &\underbrace{\frac{1}{2}(\partial_\mu\phi)^*(\partial^\mu\phi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^*\phi}_{\text{free }\phi\text{ Lagrangian}}\ \underbrace{-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}}_{\text{free }A^\mu\text{ Lagrangian}}\\ &\underbrace{+\frac{iq}{2\hbar c}\left[\phi(\partial^\mu\phi^*) - \phi^*(\partial^\mu\phi)\right]A_\mu + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2\phi^*\phi A^\mu A_\mu}_{\text{interaction term}}.\end{aligned}$$

To get the current, we apply the Euler-Lagrange equation (exploiting Eq. 10.17):

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)} = -\frac{1}{4\pi}F^{\mu\nu};\quad \frac{\delta\mathcal{L}}{\delta A_\nu} = \frac{iq}{2\hbar c}(\phi\partial^\nu\phi^* - \phi^*\partial^\nu\phi) + \left(\frac{q}{\hbar c}\right)^2|\phi|^2A^\nu.$$

So

$$-\frac{1}{4\pi}\partial_\mu F^{\mu\nu} = \frac{iq}{2\hbar c}(\phi\partial^\nu\phi^* - \phi^*\partial^\nu\phi) + \left(\frac{q}{\hbar c}\right)^2|\phi|^2A^\nu,$$

or

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\left[\frac{iq}{2\hbar}(\phi^*\partial^\nu\phi - \phi\partial^\nu\phi^*) - \frac{q^2}{\hbar^2c}|\phi|^2A^\nu\right].$$

Evidently (Eq. 10.24)

$$J^\nu = \boxed{\frac{iq}{2\hbar}(\phi^*\partial^\nu\phi - \phi\partial^\nu\phi^*) - \frac{q^2}{\hbar^2c}|\phi|^2A^\nu}.$$

(Curiously, A^ν itself appears in the last term.) And the divergence of J^μ is

$$\begin{aligned}\partial_\mu J^\mu = &\frac{iq}{2\hbar}\left[\cancel{(\partial_\mu\phi^*)(\partial^\mu\phi)} + \phi^*(\partial_\mu\partial^\mu\phi) - \cancel{(\partial_\mu\phi)(\partial^\mu\phi^*)} - \phi(\partial_\mu\partial^\mu\phi^*)\right]\\ &-\left(\frac{q^2}{\hbar^2c}\right)\left[(\partial_\mu\phi^*)\phi A^\mu + \phi^*(\partial_\mu\phi)A^\mu + \phi^*\phi(\partial_\mu A^\mu)\right].\end{aligned}$$

Meanwhile, working out the Euler–Lagrange equation for ϕ:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} &= \frac{1}{2}(\partial^\mu\phi) + \left(\frac{iq}{2\hbar c}\right)\phi A^\mu;\\ \frac{\partial\mathcal{L}}{\partial\phi^*} &= -\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi - \left(\frac{iq}{2\hbar c}\right)(\partial^\mu\phi)A_\mu + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2\phi A^\mu A_\mu.\end{aligned}$$

$$\begin{aligned}&\Rightarrow \partial_\mu\partial^\mu\phi + \left(\frac{iq}{\hbar c}\right)\left[(\partial_\mu\phi)A^\mu + \phi\partial_\mu A^\mu\right]\\ &= -\left(\frac{mc}{\hbar}\right)^2\phi - \left(\frac{iq}{\hbar c}\right)(\partial^\mu\phi)A_\mu + \left(\frac{q}{\hbar c}\right)^2\phi A^\mu A_\mu.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= \frac{1}{2}(\partial^\mu\phi)^* - \left(\frac{iq}{2\hbar c}\right)\phi^* A^\mu;\\ \frac{\partial\mathcal{L}}{\partial\phi} &= -\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^* + \left(\frac{iq}{2\hbar c}\right)(\partial^\mu\phi^*)A_\mu + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2\phi^* A^\mu A_\mu\end{aligned}$$

$$\begin{aligned}&\Rightarrow \partial_\mu\partial^\mu\phi^* - \left(\frac{iq}{\hbar c}\right)\left[(\partial_\mu\phi^*)A^\mu + \phi^*\partial_\mu A^\mu\right]\\ &= -\left(\frac{mc}{\hbar}\right)^2\phi^* + \left(\frac{iq}{\hbar c}\right)(\partial^\mu\phi^*)A_\mu + \left(\frac{q}{\hbar c}\right)^2\phi^* A^\mu A_\mu.\end{aligned}$$

Combining these results,

$$\begin{aligned}&i\left[\phi^*(\partial_\mu\partial^\mu\phi) - \phi(\partial_\mu\partial^\mu\phi^*)\right]\\ &= \frac{q}{\hbar c}\phi^*\left[(\partial_\mu\phi)A^\mu + \phi\,\partial_\mu A^\mu\right] - \cancel{i\left(\frac{mc}{\hbar}\right)^2\phi^*\phi} + \frac{q}{\hbar c}\phi^*(\partial^\mu\phi)A_\mu\\ &\quad + \cancel{\frac{iq^2}{\hbar^2c^2}\phi^*\phi A^\mu A_\mu} + \frac{q}{\hbar c}\phi\left[(\partial_\mu\phi)^*A^\mu + \phi^*\partial_\mu A^\mu\right] + \cancel{i\left(\frac{mc}{\hbar}\right)^2\phi^*\phi}\\ &\quad + \frac{q}{\hbar c}\phi(\partial^\mu\phi^*)A_\mu - \cancel{\frac{iq^2}{\hbar^2c^2}\phi^*\phi A^\mu A_\mu}\\ &= \frac{2q}{\hbar c}\left[\phi^*(\partial_\mu\phi)A^\mu + \phi(\partial_\mu\phi)^*A^\mu + \phi^*\phi\,\partial_\mu A^\mu\right].\end{aligned}$$

Thus

$$\begin{aligned}\partial_\mu J^\mu &= \frac{q}{2\hbar}\frac{2q}{\hbar c}\left[\phi^*(\partial_\mu\phi)A^\mu + \phi(\partial_\mu\phi)^*A^\mu + \phi^*\phi\,\partial_\mu A^\mu\right]\\ &\quad - \left(\frac{q^2}{\hbar^2 c}\right)\left[(\partial_\mu\phi^*)\phi A^\mu + \phi^*(\partial_\mu\phi)A^\mu + \phi^*\phi\partial_\mu A^\mu\right] = 0. \checkmark\end{aligned}$$

Problem 10.9

(a)

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi_i}\,\delta\phi_i + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\,\delta(\partial_\mu\phi_i)$$

(summation on i implied). But the Euler-Lagrange equation says

$$\frac{\partial\mathscr{L}}{\partial\phi_i} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\right), \text{ and } \delta(\partial_\mu\phi) \equiv \partial_\mu(\phi_i + \delta\phi_i) - \partial_\mu\phi_i = \partial_\mu(\delta\phi_i),$$

so

$$\delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\right)\delta\phi_i + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\delta\phi_i) = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right). \ \checkmark$$

(b) For an *infinitesimal* phase transformation, Eq. 10.26 says

$$\psi \to e^{i(\delta\theta)}\psi = (1 + i\delta\theta)\psi, \quad \text{so} \quad \delta\psi = i\psi(\delta\theta); \quad \delta\bar{\psi} = -i\bar{\psi}(\delta\theta).$$

Meanwhile (from Example 10.2),

$$\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)} = i\hbar c\bar{\psi}\gamma^\mu, \quad \frac{\partial\mathscr{L}}{\partial(\partial_\mu\bar{\psi})} = 0,$$

so

$$J^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\theta} = (i\hbar c\bar{\psi}\gamma^\mu)(i\psi) = \boxed{-\hbar c(\bar{\psi}\gamma^\mu\psi)},$$

which (apart from a constant factor $-q/\hbar$) agrees with Eq. 10.36.

(c) This time $\delta\phi = i\phi\,\delta\theta$, $\delta\phi^* = -i\phi^*\,\delta\theta$, and (Problem 10.8)

$$\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}\partial^\mu\phi^* - \frac{iq}{2\hbar c}\phi^* A^\mu, \quad \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^*)} = \frac{1}{2}\partial^\mu\phi + \frac{iq}{2\hbar c}\phi A^\mu.$$

So

$$\begin{aligned} J^\mu &= \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\theta} + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\delta\theta} \\ &= \left[\frac{1}{2}\partial^\mu\phi^* - \frac{iq}{2\hbar c}\phi^* A^\mu\right](i\phi) + \left[\frac{1}{2}\partial^\mu\phi + \frac{iq}{2\hbar c}\phi A^\mu\right](-i\phi^*) \\ &= \boxed{\frac{i}{2}\left[\phi\,\partial^\mu\phi^* - \phi^*\,\partial^\mu\phi\right] + \frac{q}{\hbar c}|\phi|^2 A^\mu}, \end{aligned}$$

which (apart from a factor $-q/\hbar$) agrees with the result in Problem 10.8.

Problem 10.10

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad H^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

Evidently H is Hermitian ($H^\dagger = H$) $\Leftrightarrow$ a and d are *real*, and $b = c^*$, four (real) numbers in all, and θ, a_1, a_2, a_3 is one way of packaging them:

$$\theta\,1 + \boldsymbol{\tau}\cdot\mathbf{a} = \theta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + a_1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + a_2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + a_3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}(\theta + a_3) & (a_1 - ia_2)\\ (a_1 + ia_2) & (\theta - a_3)\end{pmatrix},$$

which is precisely the form for the most general Hermitian 2×2 matrix, with $a = \theta + a_3$, $b = a_1 - ia_2 = c^*$, and $d = \theta - a_3$ (which, of course, can be solved to get θ and $\mathbf{a}$ in terms of a, b, c, and d).

Problem 10.11

Equations 10.56 and 10.57 $\Rightarrow$

$$\left(\partial_\mu + i\frac{q}{\hbar c}\boldsymbol{\tau}\cdot\mathbf{A}'_\mu\right)\psi' = S\left(\partial_\mu + i\frac{q}{\hbar c}\boldsymbol{\tau}\cdot\mathbf{A}_\mu\right)\psi. \qquad [1]$$

Now, $\psi' = S\psi$, so $\partial_\mu\psi' = (\partial_\mu S)\psi + S(\partial_\mu\psi)$, and Eq. [1] becomes

$$(\partial_\mu S)\psi + \cancel{S(\partial_\mu\psi)} + i\frac{q}{\hbar c}\left(\boldsymbol{\tau}\cdot\mathbf{A}'_\mu\right)S\psi = \cancel{S(\partial_\mu\psi)} + i\frac{q}{\hbar c}S\left(\boldsymbol{\tau}\cdot\mathbf{A}_\mu\right)\psi,$$

or

$$(\partial_\mu S) + i\frac{q}{\hbar c}\left(\boldsymbol{\tau}\cdot\mathbf{A}'_\mu\right)S = i\frac{q}{\hbar c}S\left(\boldsymbol{\tau}\cdot\mathbf{A}_\mu\right).$$

Multiply by S^{-1} on the right:

$$(\partial_\mu S)S^{-1} + i\frac{q}{\hbar c}\left(\boldsymbol{\tau}\cdot\mathbf{A}'_\mu\right) = i\frac{q}{\hbar c}S\left(\boldsymbol{\tau}\cdot\mathbf{A}_\mu\right)S^{-1},$$

or

$$\boldsymbol{\tau}\cdot\mathbf{A}'_\mu = S(\boldsymbol{\tau}\cdot\mathbf{A}_\mu)S^{-1} + i\frac{\hbar c}{q}\left(\partial_\mu S\right)S^{-1}. \ \checkmark$$

Problem 10.12

(a)

$$\mathbf{F}^{\mu\nu} \to \partial^\mu\left[\mathbf{A}^\nu + \partial^\nu\boldsymbol{\lambda} + \frac{2q}{\hbar c}\left(\boldsymbol{\lambda}\times\mathbf{A}^\nu\right)\right] - \partial^\nu\left[\mathbf{A}^\mu + \partial^\mu\boldsymbol{\lambda} + \frac{2q}{\hbar c}\left(\boldsymbol{\lambda}\times\mathbf{A}^\mu\right)\right]$$
$$= \partial^\mu\mathbf{A}^\nu - \partial^\nu\mathbf{A}^\mu + \partial^\mu\partial^\nu\boldsymbol{\lambda} - \partial^\nu\partial^\mu\boldsymbol{\lambda} + \frac{2q}{\hbar c}\left[\partial^\mu\left(\boldsymbol{\lambda}\times\mathbf{A}^\nu\right) - \partial^\nu\left(\boldsymbol{\lambda}\times\mathbf{A}^\mu\right)\right]$$
$$= \mathbf{F}^{\mu\nu} + \frac{2q}{\hbar c}\left[\partial^\mu\boldsymbol{\lambda}\times\mathbf{A}^\nu + \boldsymbol{\lambda}\times(\partial^\mu\mathbf{A}^\nu) - \partial^\nu\boldsymbol{\lambda}\times\mathbf{A}^\mu - \boldsymbol{\lambda}\times(\partial^\nu\mathbf{A}^\mu)\right]$$
$$= \mathbf{F}^{\mu\nu} + \frac{2q}{\hbar c}\left[\boldsymbol{\lambda}\times\mathbf{F}^{\mu\nu} + \mathbf{A}^\mu\times\partial^\nu\boldsymbol{\lambda} - \mathbf{A}^\nu\times\partial^\mu\boldsymbol{\lambda}\right].$$

(b)

$$\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu} \rightarrow \left\{\mathbf{F}^{\mu\nu} + \frac{2q}{\hbar c}\left[\boldsymbol{\lambda}\times\mathbf{F}^{\mu\nu} + \mathbf{A}^{\mu}\times\partial^{\nu}\boldsymbol{\lambda} - \mathbf{A}^{\nu}\times\partial^{\mu}\boldsymbol{\lambda}\right]\right\}$$
$$\cdot\left\{\mathbf{F}_{\mu\nu} + \frac{2q}{\hbar c}\left[\boldsymbol{\lambda}\times\mathbf{F}_{\mu\nu} + \mathbf{A}_{\mu}\times\partial_{\nu}\boldsymbol{\lambda} - \mathbf{A}_{\nu}\times\partial_{\mu}\boldsymbol{\lambda}\right]\right\}$$
$$= \mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu} + \frac{4q}{\hbar c}\left\{\mathbf{F}^{\mu\nu}\cdot\left(\boldsymbol{\lambda}\times\mathbf{F}_{\mu\nu}\right) + \mathbf{F}^{\mu\nu}\cdot\left[\left(\mathbf{A}_{\mu}\times\partial_{\nu}\boldsymbol{\lambda}\right) - \left(\mathbf{A}_{\nu}\times\partial_{\mu}\boldsymbol{\lambda}\right)\right]\right\}$$

(I dropped terms in λ^2, since we are only concerned with *infinitesimal* transformations). But $\mathbf{F}\cdot(\boldsymbol{\lambda}\times\mathbf{F}) = 0$, since the cross-product of any vector is perpendicular to the vector itself (I dropped the indices, since they are irrelevant to this argument, which holds for *each* $\mu\nu$), and

$$\mathbf{F}^{\mu\nu}\cdot\left[\left(\mathbf{A}_{\mu}\times\partial_{\nu}\boldsymbol{\lambda}\right) - \left(\mathbf{A}_{\nu}\times\partial_{\mu}\boldsymbol{\lambda}\right)\right] = -2\mathbf{F}^{\mu\nu}\cdot\left(\mathbf{A}_{\nu}\times\partial_{\mu}\boldsymbol{\lambda}\right) = 2\left(\mathbf{A}_{\nu}\times\mathbf{F}^{\mu\nu}\right)\cdot\partial_{\mu}\boldsymbol{\lambda}$$

In the first step I used $F^{\mu\nu}(t_{\mu\nu} - t_{\nu\mu}) = -2F^{\mu\nu}t_{\nu\mu}$, which holds because $F^{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$, and in the second step I used the vector identity $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = -(\mathbf{B}\times\mathbf{A})\cdot\mathbf{C}$. So

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{16\pi}\left(\frac{8q}{\hbar c}\right)\left(\mathbf{A}_{\nu}\times\mathbf{F}^{\mu\nu}\right)\cdot\partial_{\mu}\boldsymbol{\lambda}.$$

Problem 10.13

We know from Problem 10.12 that

$$\left(\partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu}\right) \rightarrow \left(\partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu}\right)$$
$$+ \left(\frac{2q}{\hbar c}\right)\left[\boldsymbol{\lambda}\times\left(\partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu}\right) + \mathbf{A}^{\mu}\times\partial^{\nu}\boldsymbol{\lambda} - \mathbf{A}^{\nu}\times\partial^{\mu}\boldsymbol{\lambda}\right]$$

So

$$\mathbf{F}^{\mu\nu} \rightarrow \left(\partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu}\right)$$
$$+ \left(\frac{2q}{\hbar c}\right)\left[\boldsymbol{\lambda}\times\left(\partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu}\right) + \mathbf{A}^{\mu}\times\partial^{\nu}\boldsymbol{\lambda} - \mathbf{A}^{\nu}\times\partial^{\mu}\boldsymbol{\lambda}\right]$$
$$- \left(\frac{2q}{\hbar c}\right)\left[\mathbf{A}^{\mu} + \partial^{\mu}\boldsymbol{\lambda} + \frac{2q}{\hbar c}\left(\boldsymbol{\lambda}\times\mathbf{A}^{\mu}\right)\right]\times\left[\mathbf{A}^{\nu} + \partial^{\nu}\boldsymbol{\lambda} + \frac{2q}{\hbar c}\left(\boldsymbol{\lambda}\times\mathbf{A}^{\nu}\right)\right]$$
$$= \mathbf{F}^{\mu\nu} + \left(\frac{2q}{\hbar c}\right)\left\{\left(\mathbf{A}^{\mu}\times\mathbf{A}^{\nu}\right) + \boldsymbol{\lambda}\times\left[\mathbf{F}^{\mu\nu} + \frac{2q}{\hbar c}\left(\mathbf{A}^{\mu}\times\mathbf{A}^{\nu}\right)\right] + \left(\mathbf{A}^{\mu}\times\partial^{\nu}\boldsymbol{\lambda}\right)\right.$$
$$- \left(\mathbf{A}^{\nu}\times\partial^{\mu}\boldsymbol{\lambda}\right) - \left(\mathbf{A}^{\mu}\times\mathbf{A}^{\nu}\right) - \left(\mathbf{A}^{\mu}\times\partial^{\nu}\boldsymbol{\lambda}\right) - \left(\partial^{\mu}\boldsymbol{\lambda}\times\mathbf{A}^{\nu}\right)$$
$$\left.- \left(\frac{2q}{\hbar c}\right)\left[\mathbf{A}^{\mu}\times\left(\boldsymbol{\lambda}\times\mathbf{A}^{\nu}\right) + \left(\boldsymbol{\lambda}\times\mathbf{A}^{\mu}\right)\times\mathbf{A}^{\nu}\right]\right\}$$

(I dropped terms of second order in λ). Three pairs of terms cancel, leaving

$$\begin{aligned}\mathbf{F}^{\mu\nu} \;\to\; & \mathbf{F}^{\mu\nu} + \left(\frac{2q}{\hbar c}\right)(\boldsymbol{\lambda} \times \mathbf{F}^{\mu\nu}) \\ & + \left(\frac{2q}{\hbar c}\right)^2 \left[\boldsymbol{\lambda} \times (\mathbf{A}^\mu \times \mathbf{A}^\nu) - \mathbf{A}^\mu \times (\boldsymbol{\lambda} \times \mathbf{A}^\nu) + (\boldsymbol{\lambda} \times \mathbf{A}^\mu) \times \mathbf{A}^\nu\right]\end{aligned}$$

The term in square brackets is zero, by virtue of the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0.$$

So

$$\mathbf{F}^{\mu\nu} \to \mathbf{F}^{\mu\nu} + \left(\frac{2q}{\hbar c}\right)(\boldsymbol{\lambda} \times \mathbf{F}^{\mu\nu}). \quad \checkmark$$

Problem 10.14

(a) From the definition (Eq. 10.65)

$$\mathbf{F}^{\mu\nu} = \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu - \frac{2q}{\hbar c}(\mathbf{A}^\mu \times \mathbf{A}^\nu)$$

we have

$$\boldsymbol{\tau} \cdot \mathbf{F}^{\mu\nu} = \partial^\mu (\boldsymbol{\tau} \cdot \mathbf{A}^\nu) - \partial^\nu (\boldsymbol{\tau} \cdot \mathbf{A}^\mu) - \frac{2q}{\hbar c}\boldsymbol{\tau} \cdot (\mathbf{A}^\mu \times \mathbf{A}^\nu).$$

Now, exploiting the result of Problem 4.20:

$$\begin{aligned}(\boldsymbol{\tau} \cdot \mathbf{A}^\mu)(\boldsymbol{\tau} \cdot \mathbf{A}^\nu) - (\boldsymbol{\tau} \cdot \mathbf{A}^\nu)(\boldsymbol{\tau} \cdot \mathbf{A}^\mu) &= A_i^\mu A_j^\nu (\tau_i \tau_j - \tau_j \tau_i) = 2i A_i^\mu A_j^\nu \epsilon_{ijk} \tau_k \\ &= 2i\boldsymbol{\tau} \cdot (\mathbf{A}^\mu \times \mathbf{A}^\nu),\end{aligned}$$

so

$$\boldsymbol{\tau} \cdot \mathbf{F}^{\mu\nu} = \partial^\mu (\boldsymbol{\tau} \cdot \mathbf{A}^\nu) - \partial^\nu (\boldsymbol{\tau} \cdot \mathbf{A}^\mu) + \frac{iq}{\hbar c}\left[(\boldsymbol{\tau} \cdot \mathbf{A}^\mu)(\boldsymbol{\tau} \cdot \mathbf{A}^\nu) - (\boldsymbol{\tau} \cdot \mathbf{A}^\nu)(\boldsymbol{\tau} \cdot \mathbf{A}^\mu)\right].$$

Invoking the transformation rule (Eq. 10.58):

$$\boldsymbol{\tau} \cdot \mathbf{A}'_\mu = S\,(\boldsymbol{\tau} \cdot \mathbf{A}_\mu)\, S^{-1} + i\frac{\hbar c}{q}(\partial_\mu S)S^{-1},$$

we have

$$\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{F}^{\mu\nu\prime} =\; & \partial^\mu \left[S\,(\boldsymbol{\tau} \cdot \mathbf{A}^\nu)\, S^{-1} + i\frac{\hbar c}{q}(\partial^\nu S)S^{-1}\right] - \partial^\nu \left[S\,(\boldsymbol{\tau} \cdot \mathbf{A}^\mu)\, S^{-1} + i\frac{\hbar c}{q}(\partial^\mu S)S^{-1}\right] \\ & + \frac{iq}{\hbar c}\left\{\left[S\,(\boldsymbol{\tau} \cdot \mathbf{A}^\mu)\, S^{-1} + i\frac{\hbar c}{q}(\partial^\mu S)S^{-1}\right]\left[S\,(\boldsymbol{\tau} \cdot \mathbf{A}^\nu)\, S^{-1} + i\frac{\hbar c}{q}(\partial^\nu S)S^{-1}\right]\right. \\ & \left. - \left[S\,(\boldsymbol{\tau} \cdot \mathbf{A}^\nu)\, S^{-1} + i\frac{\hbar c}{q}(\partial^\nu S)S^{-1}\right]\left[S\,(\boldsymbol{\tau} \cdot \mathbf{A}^\mu)\, S^{-1} + i\frac{\hbar c}{q}(\partial^\mu S)S^{-1}\right]\right\}.\end{aligned}$$

Expanding this out,

$$
\begin{aligned}
\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu\prime} =\ & (\partial^\mu S)\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,S^{-1} + S\,[\boldsymbol{\tau}\cdot(\partial^\mu\mathbf{A}^\nu)]\,S^{-1} + S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\left(\partial^\mu S^{-1}\right)\\
&\qquad + i\frac{\hbar c}{q}\partial^\mu\left[(\partial^\nu S)\,S^{-1}\right]\\
& - (\partial^\nu S)\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,S^{-1} - S\,[\boldsymbol{\tau}\cdot(\partial^\nu\mathbf{A}^\mu)]\,S^{-1} - S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\left(\partial^\nu S^{-1}\right)\\
&\qquad - i\frac{\hbar c}{q}\partial^\nu\left[(\partial^\mu S)\,S^{-1}\right]\\
& + \frac{iq}{\hbar c}\Bigg\{ S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,S^{-1}S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,S^{-1}\\
&\qquad + \frac{i\hbar c}{q}\left[S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,S^{-1}\,(\partial^\nu S)\,S^{-1} + (\partial^\mu S)\,S^{-1}S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,S^{-1}\right]\\
&\qquad - \left(\frac{\hbar c}{q}\right)^2\left[(\partial^\mu S)\,S^{-1}\,(\partial^\nu S)\,S^{-1}\right]\\
&\qquad - S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,S^{-1}S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,S^{-1}\\
&\qquad - \frac{i\hbar c}{q}\left[S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,S^{-1}\,(\partial^\mu S)\,S^{-1} + (\partial^\nu S)\,S^{-1}S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,S^{-1}\right]\\
&\qquad + \left(\frac{\hbar c}{q}\right)^2\left[(\partial^\nu S)\,S^{-1}\,(\partial^\mu S)\,S^{-1}\right]\Bigg\}
\end{aligned}
$$

The idea now is to pull out an S on the left and an S^{-1} on the right, by exploiting the hint:

$$
\partial^\mu S = -S\left(\partial^\mu S^{-1}\right)S, \qquad \partial^\mu S^{-1} = -S^{-1}\,(\partial^\mu S)\,S^{-1}.
$$

Then $\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu\prime} =$

$$
\begin{aligned}
& S\left\{\boldsymbol{\tau}\cdot(\partial^\mu\mathbf{A}^\nu) - \boldsymbol{\tau}\cdot(\partial^\nu\mathbf{A}^\mu) + \frac{iq}{\hbar c}\left[(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu) - (\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\right]\right\}S^{-1}\\
& + S\Bigg[-\left(\partial^\mu S^{-1}\right)S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu) - (\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,S^{-1}\,(\partial^\mu S) + \left(\partial^\nu S^{-1}\right)S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\\
&\qquad + (\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,S^{-1}\,(\partial^\nu S) + \left(\partial^\mu S^{-1}\right)S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\nu) - (\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\,S^{-1}\,(\partial^\nu S)\\
&\qquad + (\boldsymbol{\tau}\cdot\mathbf{A}^\nu)\,S^{-1}\,(\partial^\mu S) - \left(\partial^\nu S^{-1}\right)S\,(\boldsymbol{\tau}\cdot\mathbf{A}^\mu)\Bigg]S^{-1}\\
& + i\frac{\hbar c}{q}\left[(\partial^\mu\partial^\nu S)\,S^{-1} + (\partial^\nu S)\left(\partial^\mu S^{-1}\right) - (\partial^\nu\partial^\mu S)\,S^{-1} - (\partial^\mu S)\left(\partial^\nu S^{-1}\right)\right]\\
& - i\frac{\hbar c}{q}S\left[-\left(\partial^\mu S^{-1}\right)SS^{-1}\,(\partial^\nu S) + \left(\partial^\nu S^{-1}\right)SS^{-1}\,(\partial^\mu S)\right]S^{-1},
\end{aligned}
$$

or

$$\begin{aligned}\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu\prime} &= S\left(\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu}\right)S^{-1} + i\frac{\hbar c}{q}S\Big[\left(\partial^{\nu}S^{-1}\right)SS^{-1}\left(\partial^{\mu}S\right)\\ &\quad - \left(\partial^{\mu}S^{-1}\right)SS^{-1}\left(\partial^{\nu}S\right) + \left(\partial^{\mu}S^{-1}\right)\left(\partial^{\nu}S\right) - \left(\partial^{\nu}S^{-1}\right)\left(\partial^{\mu}S\right)\Big]S^{-1}\\ &= S\left(\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu}\right)S^{-1}. \quad \checkmark\end{aligned}$$

(b)

$$\begin{aligned}\mathrm{Tr}\left[\left(\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu\prime}\right)\left(\boldsymbol{\tau}\cdot\mathbf{F}_{\mu\nu}{}'\right)\right] &= \mathrm{Tr}\left[\left(S\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu}S^{-1}\right)\left(S\boldsymbol{\tau}\cdot\mathbf{F}_{\mu\nu}S^{-1}\right)\right]\\ &= \mathrm{Tr}\left[S^{-1}\left(S\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu}\right)\left(\boldsymbol{\tau}\cdot\mathbf{F}_{\mu\nu}\right)\right] = \mathrm{Tr}\left[\left(\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu}\right)\left(\boldsymbol{\tau}\cdot\mathbf{F}_{\mu\nu}\right)\right]. \quad \checkmark\end{aligned}$$

(c)

$$\begin{aligned}\mathrm{Tr}\left[\left(\boldsymbol{\tau}\cdot\mathbf{F}^{\mu\nu}\right)\left(\boldsymbol{\tau}\cdot\mathbf{F}_{\mu\nu}\right)\right] &= F_i^{\mu\nu}F_{\mu\nu_j}\mathrm{Tr}(\tau_i\tau_j) = F_i^{\mu\nu}F_{\mu\nu_j}\mathrm{Tr}\left(\delta_{ij} + i\epsilon_{ijk}\tau_k\right)\\ &= F_i^{\mu\nu}F_{\mu\nu_j}\delta_{ij}\mathrm{Tr}\left(1\right) = 2\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}.\end{aligned}$$

(I used the fact that the Pauli matrices are traceless, and the trace of the 2×2 unit matrix is 2.) *Conclusion:* the Lagrangian in Eq. 10.63,

$$\mathcal{L} = -\frac{1}{16\pi}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu},$$

is invariant under *finite* gauge transformations. [But since a finite transformation can be built up from infinitesimal transformations, it is really unnecessary to check anything beyond the infinitesimal case, which is typically much easier.] ✓

Problem 10.15

According to Eq. 10.65,

$$\mathbf{F}^{\mu\nu} \equiv \partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu} - \left(\frac{2q}{\hbar c}\right)\left(\mathbf{A}^{\mu}\times\mathbf{A}^{\nu}\right).$$

Putting this into the Lagrangian (Eq.10.69):

$$\begin{aligned}\mathcal{L} &= -\frac{1}{16\pi}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \frac{1}{c}\mathbf{J}^{\mu}\cdot\mathbf{A}_{\mu} = -\frac{1}{16\pi}\Bigg\{\left(\partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu}\right)\cdot\left(\partial_{\mu}\mathbf{A}_{\nu} - \partial_{\nu}\mathbf{A}_{\mu}\right)\\ &\quad - 2\left(\frac{2q}{\hbar c}\right)\left(\partial_{\mu}\mathbf{A}_{\nu} - \partial_{\nu}\mathbf{A}_{\mu}\right)\cdot\left(\mathbf{A}^{\mu}\times\mathbf{A}^{\nu}\right) + \left(\frac{2q}{\hbar c}\right)^2\left(\mathbf{A}^{\mu}\times\mathbf{A}^{\nu}\right)\cdot\left(\mathbf{A}_{\mu}\times\mathbf{A}_{\nu}\right)\Bigg\}\\ &\quad - \frac{1}{c}\mathbf{J}^{\mu}\cdot\mathbf{A}_{\mu}.\end{aligned}$$

Referring to Eq. 10.17 (if this notation—with a 3-vector in the denominator—disturbs you, write it out in component form):

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbf{A}_\nu)} = -\frac{1}{4\pi}\left(\partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu\right) + \frac{1}{16\pi}2\left(\frac{2q}{\hbar c}\right)2\left(\mathbf{A}^\mu \times \mathbf{A}^\nu\right) = -\frac{1}{4\pi}\mathbf{F}^{\mu\nu}.$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}_\nu} = -\frac{1}{16\pi}\left[2\left(\frac{2q}{\hbar c}\right)2\mathbf{A}_\mu \times \left(\partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu\right) - \left(\frac{2q}{\hbar c}\right)^2 4\mathbf{A}_\mu \times \left(\mathbf{A}^\mu \times \mathbf{A}^\nu\right)\right]$$
$$-\frac{1}{c}\mathbf{J}^\nu = -\frac{1}{4\pi}\left(\frac{2q}{\hbar c}\right)\mathbf{A}_\mu \times \mathbf{F}^{\mu\nu} - \frac{1}{c}\mathbf{J}^\nu.$$

To isolate $\mathbf{A}_\nu$ for the derivatives, I used the vector identity $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$. The Euler-Lagrange equation is

$$\boxed{\partial_\mu \mathbf{F}^{\mu\nu} = \left(\frac{2q}{\hbar c}\right)\mathbf{A}_\mu \times \mathbf{F}^{\mu\nu} + \frac{4\pi}{c}\mathbf{J}^\nu}.$$

Before we cast this in the more familiar nonrelativistic form it is best to express the 3-vectors in index form (we will need the bold face for *spatial* 3-vectors):

$$\partial_\mu F_i^{\mu\nu} = \left(\frac{2q}{\hbar c}\right)\epsilon_{ijk} A_{j_\mu} F_k^{\mu\nu} + \frac{4\pi}{c}J_i^\nu,$$

where i runs from 1 to 3, and summation over j and k is implied. With $\nu = 0$ we have (in the notation of Eqs. 7.71, 7.72, and 7.79):

$$\boxed{\nabla\cdot\mathbf{E}_i = -\left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}\mathbf{A}_j\cdot\mathbf{E}_k + 4\pi\rho_i}.$$

For $\nu = 1, 2,$ and 3:

$$\boxed{\nabla\times\mathbf{B}_i - \frac{1}{c}\frac{\partial \mathbf{E}_i}{\partial t} = -\left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}\left(V_j\mathbf{E}_k + \mathbf{A}_j\times\mathbf{B}_k\right) + \frac{4\pi}{c}\mathbf{J}_i}.$$

These are the *in*homogeneous "Maxwell" equations; the homogeneous equations come from the definition of $\mathbf{F}^{\mu\nu}$:

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu - \left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}A_j^\mu A_k^\nu.$$

For $\mu = 0$ we get

$$\mathbf{E}_i = -\nabla V_i - \frac{1}{c}\frac{\partial \mathbf{A}_i}{\partial t} + \left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}V_j\mathbf{A}_k.$$

For $\mu = 1, \nu = 2$:

$$\begin{aligned} F_i^{12} &= \partial^1 A_i^2 - \partial^2 A_i^1 - \left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}A_j^1 A_k^2 \\ -B_{iz} &= -\nabla_x A_{iy} + \nabla_y A_{ix} - \left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}A_{j_x}A_{ky}. \end{aligned}$$

Now

$$\epsilon_{ijk}A_{j_x}A_{ky} = \epsilon_{ikj}A_{kx}A_{j_y} = -\epsilon_{ijk}A_{j_y}A_{kx},$$

so

$$\epsilon_{ijk}A_{j_x}A_{ky} = \tfrac{1}{2}\epsilon_{ijk}[A_{j_x}A_{ky} - A_{j_y}A_{kx}] = \tfrac{1}{2}\epsilon_{ijk}\left(\mathbf{A}_j \times \mathbf{A}_k\right)_z.$$

Evidently

$$\mathbf{B}_i = \nabla \times \mathbf{A}_i + \tfrac{1}{2}\left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}\left(\mathbf{A}_j \times \mathbf{A}_k\right).$$

Hence

$$\boxed{\nabla \cdot \mathbf{B}_i = \left(\frac{q}{\hbar c}\right)\epsilon_{ijk}\nabla \cdot \left(\mathbf{A}_j \times \mathbf{A}_k\right)}.$$

Meanwhile,

$$\nabla \times \mathbf{E}_i = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{A}_i) + \left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}\nabla \times (V_j \mathbf{A}_k).$$

But

$$\nabla \times \mathbf{A}_i = \mathbf{B}_i - \tfrac{1}{2}\left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}\left(\mathbf{A}_j \times \mathbf{A}_k\right),$$

so

$$\boxed{\nabla \times \mathbf{E}_i = -\frac{1}{c}\frac{\partial \mathbf{B}_i}{\partial t} + \left(\frac{2q}{\hbar c}\right)\epsilon_{ijk}\left[\frac{1}{2c}\frac{\partial}{\partial t}\left(\mathbf{A}_j \times \mathbf{A}_k\right) + \nabla \times (V_j \mathbf{A}_k)\right]}.$$

Problem 10.16

This is the 3×3 analog to Problem 10.10.

$$H = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}; \quad H^\dagger = \begin{pmatrix} a^* & d^* & g^* \\ b^* & e^* & h^* \\ c^* & f^* & j^* \end{pmatrix}.$$

Evidently H is Hermitian ($H^\dagger = H$) $\Leftrightarrow$ a, e, and j are *real*, and $b = d^*$, $c = g^*$, $f = h^*$, nine (real) numbers in all, and $\theta, a_1, a_2, \ldots, a_8$ is one way of packaging them:

$$\theta 1 + \boldsymbol{\lambda} \cdot \mathbf{a} = \theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ a_4 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} + a_6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + a_7 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$
$$+ a_8 \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} (\theta + a_3 + a_8/\sqrt{3}) & (a_1 - ia_2) & (a_4 - ia_5) \\ (a_1 + ia_2) & (\theta - a_3 + a_8/\sqrt{3}) & a_6 - ia_7) \\ (a_4 + ia_5) & (a_6 + ia_7) & (\theta - 2a_8/\sqrt{3}) \end{pmatrix}.$$

This is precisely the form for the most general Hermitian 3×3 matrix, with $a = (\theta + a_3 + a_8/\sqrt{3})$, $b = a_1 - ia_2 = d^*$, $c = a_4 - ia_5 = g^*$, $e = (\theta - a_3 + a_8/\sqrt{3})$, $f = a_6 - ia_7 = h^*$, and $j = (\theta - 2a_8/\sqrt{3})$ (which, of course, can be solved to get θ and $\mathbf{a}$ in terms of a, b, c, d, e, f, g, h, and j).

Problem 10.17

(a) First suppose that A is *diagonal*:

$$A = D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}.$$

Then

$$D^j = \begin{pmatrix} d_1^j & 0 & 0 & \cdots & 0 \\ 0 & d_2^j & 0 & \cdots & 0 \\ 0 & 0 & d_3^j & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^j \end{pmatrix},$$

and hence

$$
\begin{aligned}
e^D &= 1 + D + \frac{1}{2}D^2 + \frac{1}{3!}D^3 + \cdots \\
&= \begin{pmatrix}
(1 + d_1 + \frac{1}{2}d_1^2 \cdots) & 0 & \cdots & 0 \\
0 & (1 + d_2 + \frac{1}{2}d_2^2 + \cdots) & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & (1 + d_n + \frac{1}{2}d_n^2 + \cdots)
\end{pmatrix} \\
&= \begin{pmatrix}
e^{d_1} & 0 & \cdots & 0 \\
0 & e^{d_2} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & e^{d_n}
\end{pmatrix}.
\end{aligned}
$$

Thus

$$\det\left(e^D\right) = e^{d_1}e^{d_2}e^{d_3}\cdots e^{d_n} = e^{d_1+d_2+d_3+\cdots+d_n} = e^{\mathrm{Tr}(D)}. \checkmark$$

Now suppose that A is *diagonalizable*:

$$S^{-1}AS = D = \begin{pmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & d_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & d_n
\end{pmatrix}.$$

Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, it follows that

$$\mathrm{Tr}(D) = \mathrm{Tr}(S^{-1}AS) = \mathrm{Tr}(SS^{-1}A) = \mathrm{Tr}(A),$$

and (inserting a factor of $SS^{-1} = 1$ between adjacent A's)

$$S^{-1}A^2S = (S^{-1}AS)(S^{-1}AS) = D^2,$$

$$S^{-1}A^3S = (S^{-1}AS)(S^{-1}AS)(S^{-1}AS) = D^3,$$

etc. So

$$
\begin{aligned}
S^{-1}e^AS &= S^{-1}\left(1 + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots\right)S \\
&= 1 + D + \frac{1}{2}D^2 + \frac{1}{3!}D^3 + \cdots = e^D. \qquad \bigstar
\end{aligned}
$$

But $\det(AB) = \det(A)\det(B)$, so

$$
\begin{aligned}
\det\left(S^{-1}e^AS\right) &= [\det(S^{-1})]\left[\det\left(e^A\right)\right][\det(S)] = [\det(S^{-1}S)]\left[\det\left(e^A\right)\right] \\
&= \det\left(e^A\right).
\end{aligned}
$$

Conclusion:

$$\det\left(e^A\right) = \det\left(S^{-1}e^AS\right) = \det\left(e^D\right) = e^{\mathrm{Tr}(D)} = e^{\mathrm{Tr}(A)}. \checkmark$$

Finally, the general case. Because $S^{-1}AS = J$, it follows from the same argument as before (★) that

$$S^{-1}e^AS = e^J, \quad \text{and hence} \quad \det(e^A) = \det(e^J).$$

Now J has only zeroes above the main diagonal:

$$J = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ x & d_2 & 0 & \cdots & 0 \\ x & x & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & \cdots & d_n \end{pmatrix},$$

(where the x's stand for possibly nonzero elements), and this feature persists for powers of J:

$$J^j = \begin{pmatrix} d_1^j & 0 & 0 & \cdots & 0 \\ x & d_2^j & 0 & \cdots & 0 \\ x & x & d_3^j & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & \cdots & d_n^j \end{pmatrix},$$

so

$$e^J = \begin{pmatrix} e^{d_1} & 0 & 0 & \cdots & 0 \\ x & e^{d_2} & 0 & \cdots & 0 \\ x & x & e^{d_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & \cdots & e^{d_n} \end{pmatrix}.$$

Evaluating each determinant by minors along the top row:

$$\det\left(e^J\right) = e^{d_1}\begin{pmatrix} e^{d_2} & 0 & \cdots & 0 \\ x & e^{d_3} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x & x & \cdots & e^{d_n} \end{pmatrix} = e^{d_1}e^{d_2}\begin{pmatrix} e^{d_3} & \cdots & 0 \\ \vdots & & \vdots \\ x & \cdots & e^{d_n} \end{pmatrix}$$
$$= e^{d_1}e^{d_2}e^{d_3}\cdots e^{d_n} = e^{d_1+d_2+d_3+\cdots+d_n} = e^{\mathrm{Tr}(J)}.$$

But $\mathrm{Tr}(J) = \mathrm{Tr}(S^{-1}AS) = \mathrm{Tr}(A)$, so

$$\det\left(e^A\right) = \det\left(e^J\right) = e^{\mathrm{Tr}(J)} = e^{\mathrm{Tr}(A)}. \checkmark$$

(b) From **(a)** we know that

$$\det\left(e^{i\boldsymbol{\lambda}\cdot\mathbf{a}}\right) = e^{\mathrm{Tr}(i\boldsymbol{\lambda}\cdot\mathbf{a})}.$$

But (Problem 10.16)

$$\mathrm{Tr}(i\boldsymbol{\lambda}\cdot\mathbf{a}) = i\left[\left(a_3 + \frac{a_8}{\sqrt{3}}\right) + \left(-a_3 + \frac{a_8}{\sqrt{3}}\right) + \left(-2\frac{a_8}{\sqrt{3}}\right)\right] = 0.$$

So

$$e^{\mathrm{Tr}(i\boldsymbol{\lambda}\cdot\mathbf{a})} = e^0 = 1, \quad \text{and hence} \quad \det\left(e^{i\boldsymbol{\lambda}\cdot\mathbf{a}}\right) = 1. \checkmark$$

Problem 10.18

The first part is identical to Problem 10.11, with $\boldsymbol{\tau}\to\boldsymbol{\lambda}$. Equations 10.80 and 10.81 $\Rightarrow$

$$\left(\partial_\mu + i\frac{q}{\hbar c}\boldsymbol{\lambda}\cdot\mathbf{A}'_\mu\right)\psi' = S\left(\partial_\mu + i\frac{q}{\hbar c}\boldsymbol{\lambda}\cdot\mathbf{A}_\mu\right)\psi. \qquad [1]$$

Now, $\psi' = S\psi$, so $\partial_\mu\psi' = (\partial_\mu S)\psi + S(\partial_\mu\psi)$, and Eq. [1] becomes

$$(\partial_\mu S)\psi + \cancel{S(\partial_\mu\psi)} + i\frac{q}{\hbar c}\left(\boldsymbol{\lambda}\cdot\mathbf{A}'_\mu\right)S\psi = \cancel{S(\partial_\mu\psi)} + i\frac{q}{\hbar c}S\left(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu\right)\psi,$$

or

$$(\partial_\mu S) + i\frac{q}{\hbar c}\left(\boldsymbol{\lambda}\cdot\mathbf{A}'_\mu\right)S = i\frac{q}{\hbar c}S\left(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu\right).$$

Multiply by S^{-1} on the right:

$$(\partial_\mu S)S^{-1} + i\frac{q}{\hbar c}\left(\boldsymbol{\lambda}\cdot\mathbf{A}'_\mu\right) = i\frac{q}{\hbar c}S\left(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu\right)S^{-1},$$

or

$$\boldsymbol{\lambda}\cdot\mathbf{A}'_\mu = S(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu)S^{-1} + i\frac{\hbar c}{q}\left(\partial_\mu S\right)S^{-1}. \checkmark \qquad \text{(Eq. 10.82)}$$

The second part is the same as Eqs. 10.58-10.61, only with $\boldsymbol{\tau}\to\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}\to\boldsymbol{\phi}$. The infinitesimal transformations (up to first order in $\boldsymbol{\phi}$) are:

$$S = e^{-iq\boldsymbol{\lambda}\cdot\boldsymbol{\phi}/\hbar c} \cong 1 - \frac{iq}{\hbar c}\boldsymbol{\lambda}\cdot\boldsymbol{\phi}, \quad S^{-1} \cong 1 + \frac{iq}{\hbar c}\boldsymbol{\lambda}\cdot\boldsymbol{\phi}, \quad \partial_\mu S \cong -\frac{iq}{\hbar c}\boldsymbol{\lambda}\cdot\left(\partial_\mu\boldsymbol{\phi}\right)$$

In this approximation Eq. 10.82 becomes

$$\begin{aligned}\boldsymbol{\lambda}\cdot\mathbf{A}'_\mu &\cong \left(1 - \frac{iq}{\hbar c}\boldsymbol{\lambda}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu\right)\left(1 + \frac{iq}{\hbar c}\boldsymbol{\lambda}\cdot\boldsymbol{\phi}\right)\\ &\quad + i\frac{\hbar c}{q}\left[-\frac{iq}{\hbar c}\boldsymbol{\lambda}\cdot\left(\partial_\mu\boldsymbol{\phi}\right)\right]\left(1 + \frac{iq}{\hbar c}\boldsymbol{\lambda}\cdot\boldsymbol{\phi}\right)\\ &\cong \boldsymbol{\lambda}\cdot\mathbf{A}_\mu + \frac{iq}{\hbar c}\left[(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu)(\boldsymbol{\lambda}\cdot\boldsymbol{\phi}) - (\boldsymbol{\lambda}\cdot\boldsymbol{\phi})(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu)\right] + \boldsymbol{\lambda}\cdot\left(\partial_\mu\boldsymbol{\phi}\right)\end{aligned}$$

The term in square brackets is a commutator; using Eq. 8.35:

$$\left[(\boldsymbol{\lambda}\cdot\mathbf{A}_\mu),(\boldsymbol{\lambda}\cdot\boldsymbol{\phi})\right] = A_\mu^\alpha\phi^\beta\left[\lambda^\alpha,\lambda^\beta\right] = A_\mu^\alpha\phi^\beta\left(2if^{\alpha\beta\gamma}\lambda^\gamma\right) = -2i\boldsymbol{\lambda}\cdot(\boldsymbol{\phi}\times\mathbf{A}_\mu)$$

(summation on α, β, and γ from 1 to 8 implied; in the last step I adopted the generalized dot- and cross-product notation introduced in the text—see Eq. 10.84). *Conclusion:*

$$\boldsymbol{\lambda}\cdot\mathbf{A}'_\mu \cong \boldsymbol{\lambda}\cdot\mathbf{A}_\mu + \boldsymbol{\lambda}\cdot(\partial_\mu\boldsymbol{\phi}) + \left(\frac{2q}{\hbar c}\right)\boldsymbol{\lambda}\cdot(\boldsymbol{\phi}\times\mathbf{A}_\mu)]$$

or

$$\mathbf{A}'_\mu \cong \mathbf{A}_\mu + (\partial_\mu\boldsymbol{\phi}) + \left(\frac{2q}{\hbar c}\right)(\boldsymbol{\phi}\times\mathbf{A}_\mu)\,. \quad \checkmark \qquad \text{(Eq. 10.83)}$$

Problem 10.19

In the notation suggested on page 370:

$$T_{\mu\nu} = \left[-p^2+(mc)^2\right]g_{\mu\nu} + p_\mu p_\nu \qquad \text{(Eq. 10.92)}$$

$$(T^{-1})_{\mu\nu} = \frac{-1}{p^2-(mc)^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{(mc)^2}\right) \qquad \text{(Eq. 10.95)}$$

$$\begin{aligned}
T_{\mu\lambda}(T^{-1})^{\lambda\nu} &= \left[\left(-p^2+(mc)^2\right)g_{\mu\lambda} + p_\mu p_\lambda\right]\frac{-1}{p^2-(mc)^2}\left(g^{\lambda\nu} - \frac{p^\lambda p^\nu}{(mc)^2}\right)\\
&= g_{\mu\lambda}\left(g^{\lambda\nu} - \frac{p^\lambda p^\nu}{(mc)^2}\right) - \frac{p_\mu p_\lambda}{p^2-(mc)^2}\left(g^{\lambda\nu} - \frac{p^\lambda p^\nu}{(mc)^2}\right)\\
&= \delta_\mu^\nu - \frac{p_\mu p^\nu}{(mc)^2} - \frac{p_\mu p^\nu}{p^2-(mc)^2} + \frac{p_\mu p^\nu p^2}{(mc)^2[p^2-(mc)^2]}\\
&= \delta_\mu^\nu + \frac{p_\mu p^\nu}{(mc)^2\left[p^2-(mc)^2\right]}\left[-p^2+(mc)^2-(mc)^2+p^2\right] = \delta_\mu^\nu. \quad \checkmark
\end{aligned}$$

Problem 10.20

There are three free Klein-Gordon Lagrangians (one for each particle type), and one interaction term, of the form $i\mathscr{L} = -ig\phi_A\phi_B\phi_C$ (since the vertex factor is $-ig$):

$$\begin{aligned}
\mathscr{L} &= \left[\frac{1}{2}(\partial_\mu\phi_A)(\partial^\mu\phi_A) - \frac{1}{2}\left(\frac{m_A c}{\hbar}\right)^2\phi_A^2\right] + \left[\frac{1}{2}(\partial_\mu\phi_B)(\partial^\mu\phi_B) - \frac{1}{2}\left(\frac{m_B c}{\hbar}\right)^2\phi_B^2\right]\\
&\quad + \left[\frac{1}{2}(\partial_\mu\phi_C)(\partial^\mu\phi_C) - \frac{1}{2}\left(\frac{m_C c}{\hbar}\right)^2\phi_C^2\right] - g\phi_A\phi_B\phi_C.
\end{aligned}$$

[Actually, for $\mathcal{L}_{\text{int}}$ to have the correct dimensions (energy/volume—p. 357), with g carrying the units of momentum (p. 213), when the scalar fields have the dimensions given on p. 357, we must have $\mathcal{L}_{\text{int}} = -(1/\hbar\sqrt{\hbar c})g\phi_A\phi_B\phi_C$, and now the naive construction of the vertex factor does not get the c's and $\hbar$'s right. See remark before Eq. 10.101.]

Problem 10.21

This describes a theory in which there are three kinds of particles:

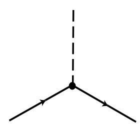
- a spin-$\frac{1}{2}$ particle of mass m_1,
- its antiparticle (likewise mass m_1),

for both of which the propagator is $i\dfrac{\not{q} + m_1 c}{q^2 - (m_1 c)^2}$, and

- a spin-0 particle of mass m_2,

for which the propagator is $\dfrac{i}{p^2 - (m_2 c)^2}$.

The primitive vertex is



Vertex factor: $\boxed{i\alpha_Y}$.

Problem 10.22

Equation 10.119 says $\phi_1 = \eta + (\mu/\lambda)$, $\phi_2 = \xi$, so

$$\partial_\mu\phi_1 = \partial_\mu\eta, \quad \partial_\mu\phi_2 = \partial_\mu\xi, \quad \phi_1^2 + \phi_2^2 = \eta^2 + 2\eta\frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2} + \xi^2,$$

Substitute these into the Lagrangian (Eq. 10.115):

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{2}(\partial_\mu \eta)\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \xi)\partial^\mu \xi) + \frac{1}{2}\mu^2 \left(\eta^2 + 2\eta\frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2} + \xi^2\right) \\
&\quad - \frac{1}{4}\lambda^2 \left(\eta^2 + 2\eta\frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2} + \xi^2\right)^2 \\
&= \frac{1}{2}\left[(\partial_\mu \eta)\partial^\mu \eta) + (\partial_\mu \xi)\partial^\mu \xi)\right] + \frac{1}{2}\mu^2\eta^2 + \eta\frac{\mu^3}{\lambda} + \frac{1}{2}\frac{\mu^4}{\lambda^2} + \frac{1}{2}\mu^2\xi^2 \\
&\quad - \frac{1}{4}\lambda^2\left[\eta^4 + 4\eta^2\frac{\mu^2}{\lambda^2} + \frac{\mu^4}{\lambda^4} + \xi^4 + 4\eta^3\frac{\mu}{\lambda} + 2\eta^2\frac{\mu^2}{\lambda^2} + 2\eta^2\xi^2\right. \\
&\qquad\qquad \left. + 4\eta\frac{\mu^3}{\lambda^3} + 4\eta\xi^2\frac{\mu}{\lambda} + 2\xi^2\frac{\mu^2}{\lambda^2}\right] \\
&= \frac{1}{2}\left[(\partial_\mu \eta)\partial^\mu \eta) + (\partial_\mu \xi)\partial^\mu \xi)\right] + \eta^2\mu^2\left(\frac{1}{2} - 1 - \frac{1}{2}\right) + \xi^2\mu^2\left(\frac{1}{2} - \frac{1}{2}\right) \\
&\quad + \frac{\mu^4}{\lambda^2}\left(\frac{1}{2} - \frac{1}{4}\right) + \eta\frac{\mu^3}{\lambda}(1-1) - \frac{\lambda^2}{4}\left(\eta^4 + \xi^4 + 2\eta^2\xi^2\right) \\
&\quad - \lambda\mu(\eta^3 + \eta\xi^2) + \frac{\mu^4}{\lambda^2}\left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi)\right] - \frac{\lambda^2}{4}\left(\eta^4 + \xi^4 + 2\eta^2\xi^2\right) \\
&\quad - \lambda\mu(\eta^3 + \eta\xi^2) + \frac{\mu^4}{4\lambda^2}. \ \checkmark
\end{aligned}
$$

Problem 10.23

$$\psi_1 + \psi_2 = \sqrt{2}\,\eta \ \Rightarrow\ \eta = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2); \quad \psi_1 - \psi_2 = \sqrt{2}\,\xi \ \Rightarrow\ \xi = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$$

So

$$
\begin{aligned}
(\partial_\mu \eta)(\partial^\mu \eta) \ &+\ (\partial_\mu \xi)(\partial^\mu \xi) \\
&= \frac{1}{2}\left[(\partial_\mu\psi_1 + \partial_\mu\psi_2)(\partial^\mu\psi_1 + \partial^\mu\psi_2) + (\partial_\mu\psi_1 - \partial_\mu\psi_2)(\partial^\mu\psi_1 - \partial^\mu\psi_2)\right] \\
&= (\partial_\mu\psi_1)(\partial^\mu\psi_1) + (\partial_\mu\psi_2)(\partial^\mu\psi_2);
\end{aligned}
$$

$$(\eta^2 + \xi^2) = \frac{1}{2}\left[(\psi_1 + \psi_2)^2 + (\psi_1 - \psi_2)^2\right] = \psi_1^2 + \psi_2^2.$$

Putting this into Eq. 10.120:

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}\left[(\partial_\mu\psi_1)(\partial^\mu\psi_1) + (\partial_\mu\psi_2)(\partial^\mu\psi_2)\right] - \frac{\mu^2}{2}(\psi_1+\psi_2)^2 \\ &\quad - \frac{\mu\lambda}{2\sqrt{2}}(\psi_1+\psi_2)(\psi_1^2+\psi_2^2) - \frac{\lambda^4}{16}(\psi_1^2+\psi_2^2)^2 + \frac{\mu^4}{4\lambda^2} \\ &= \left[\frac{1}{2}(\partial_\mu\psi_1)(\partial^\mu\psi_1) - \frac{\mu^2}{2}\psi_1^2\right] + \left[\frac{1}{2}(\partial_\mu\psi_2)(\partial^\mu\psi_2) - \frac{\mu^2}{2}\psi_2^2\right] - \mu^2\psi_1\psi_2 \\ &\quad - \frac{\mu\lambda}{2\sqrt{2}}\left(\psi_1^3 + \psi_1\psi_2^2 + \psi_2\psi_1^2 + \psi_2^3\right) - \frac{\lambda^4}{16}\left(\psi_1^4 + \psi_2^4 + 2\psi_1^2\psi_2^2\right) + \frac{\mu^4}{4\lambda^2}.\end{aligned}$$

Problem 10.24

$$\begin{aligned}\mathscr{L} &= \tfrac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \tfrac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) + \tfrac{1}{2}(\partial_\mu\phi_3)(\partial^\mu\phi_3) \\ &\quad + \tfrac{1}{2}\mu^2(\phi_1^2+\phi_2^2+\phi_3^2) - \tfrac{1}{4}\lambda^2(\phi_1^2+\phi_2^2+\phi_3^2)^2\end{aligned}$$

This is invariant under rotations in ϕ_1, ϕ_2, ϕ_3 space. The "potential energy" function is

$$\mathscr{U} = -\tfrac{1}{2}\mu^2(\phi_1^2+\phi_2^2+\phi_3^2) + \tfrac{1}{4}\lambda^2(\phi_1^2+\phi_2^2+\phi_3^2)^2$$

and the minima lie on a *sphere* of radius μ/λ:

$$\phi_{1_{\text{min}}}^2 + \phi_{2_{\text{min}}}^2 + \phi_{3_{\text{min}}}^2 = \mu^2/\lambda^2$$

To apply the Feynman calculus, we expand about a *particular* ground state ("the vacuum")—we may as well pick

$$\phi_{1_{\text{min}}} = \mu/\lambda, \quad \phi_{2_{\text{min}}} = 0, \quad \phi_{3_{\text{min}}} = 0.$$

As before, we introduce new fields, η, ξ, and ζ, which are the fluctuations about this vacuum state:

$$\eta \equiv \phi_1 - \mu/\lambda, \quad \xi \equiv \phi_2, \quad \zeta \equiv \phi_3$$

Rewriting the Lagrangian in terms of these new field variables, we find :

$$\begin{aligned}\mathscr{L} &= \left[\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)\right] + \left[\frac{1}{2}(\partial_\mu\zeta)(\partial^\mu\zeta)\right] \\ &\quad - \left\{\mu\lambda(\eta^3 + \eta\xi^2 + \eta\zeta^2) + \frac{\lambda^2}{4}\left[\eta^4 + \xi^4 + \zeta^4 + 2\left(\eta^2\xi^2 + \eta^2\zeta^2 + \xi^2\zeta^2\right)\right]\right\} \\ &\quad + \frac{\mu^4}{4\lambda^2}.\end{aligned}$$

The first term is a free Klein–Gordon Lagrangian (Eq. 10.11) for the field η, which evidently carries a mass $\boxed{m_\eta = \sqrt{2}\,\mu\hbar/c}$; the second and third terms are free Lagrangians for the fields ξ and ζ, which are massless: $\boxed{m_\xi = m_\zeta = 0}$; and the fourth term defines various couplings. Evidently in this theory there are $\boxed{\text{2 Goldstone bosons}}$.

Problem 10.25

The last term in Eq. 10.129, $-(1/16\pi)F^{\mu\nu}F_{\mu\nu}$, is not altered by the transformation in Eq. 10.130. The previous two terms are the same as those in Eq. 10.115, and we have already shown (Problem 10.22) that they become

$$-\mu^2\eta^2 - \frac{\lambda^2}{4}\left(\eta^4 + \xi^4 + 2\eta^2\xi^2\right) - \lambda\mu(\eta^3 + \eta\xi^2) + \frac{\mu^4}{4\lambda^2}.$$

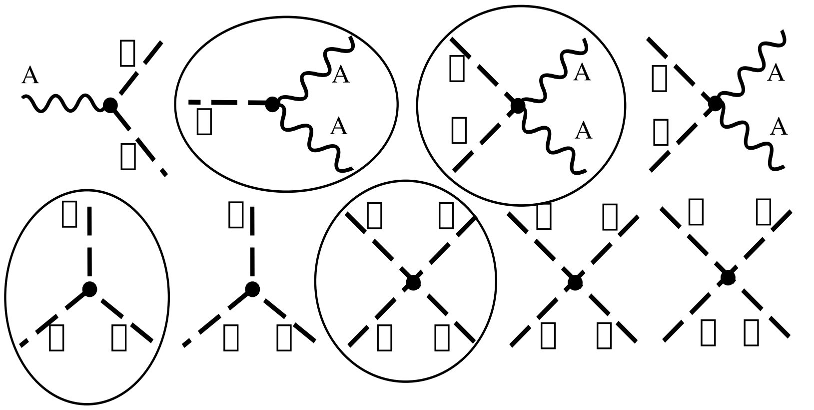
It remains only to work out the first term:

$$\begin{aligned}
&\left[\left(\partial_\mu - \frac{iq}{\hbar c}A_\mu\right)\phi^*\right]\left[\left(\partial^\mu + \frac{iq}{\hbar c}A^\mu\right)\phi\right] \\
&= \left[\partial_\mu\phi_1 - i\partial_\mu\phi_2 - \frac{iq}{\hbar c}A_\mu\phi_1 - \frac{q}{\hbar c}A_\mu\phi_2\right]\left[\partial^\mu\phi_1 + i\partial^\mu\phi_2 + \frac{iq}{\hbar c}A^\mu\phi_1 - \frac{q}{\hbar c}A^\mu\phi_2\right] \\
&= (\partial_\mu\phi_1)(\partial^\mu\phi_1) + \cancel{i(\partial_\mu\phi_1)(\partial^\mu\phi_2)} + \cancel{\frac{iq}{\hbar c}(\partial_\mu\phi_1)A^\mu\phi_1} - \frac{q}{\hbar c}(\partial_\mu\phi_1)A^\mu\phi_2 \\
&\quad - \cancel{i(\partial_\mu\phi_2)(\partial^\mu\phi_1)} + (\partial_\mu\phi_2)(\partial^\mu\phi_2) + \frac{q}{\hbar c}(\partial_\mu\phi_2)A^\mu\phi_1 + \cancel{\frac{iq}{\hbar c}(\partial_\mu\phi_2)A^\mu\phi_2} \\
&\quad - \cancel{\frac{iq}{\hbar c}(\partial^\mu\phi_1)A_\mu\phi_1} + \frac{q}{\hbar c}(\partial^\mu\phi_2)A_\mu\phi_1 + \left(\frac{q}{\hbar c}\right)^2\phi_1\phi_1A_\mu A^\mu + \cancel{i\left(\frac{q}{\hbar c}\right)^2\phi_1\phi_2A_\mu A^\mu} \\
&\quad - \frac{q}{\hbar c}(\partial^\mu\phi_1)A_\mu\phi_2 - \cancel{\frac{iq}{\hbar c}(\partial^\mu\phi_2)A_\mu\phi_2} - \cancel{i\left(\frac{q}{\hbar c}\right)^2\phi_1\phi_2A_\mu A^\mu} + \left(\frac{q}{\hbar c}\right)^2\phi_2\phi_2A_\mu A^\mu \\
&= (\partial_\mu\phi_1)(\partial^\mu\phi_1) + (\partial_\mu\phi_2)(\partial^\mu\phi_2) + \frac{2q}{\hbar c}\left[-(\partial_\mu\phi_1)\phi_2 + (\partial_\mu\phi_2)\phi_1\right]A^\mu \\
&\quad + \left(\frac{q}{\hbar c}\right)^2(\phi_1^2 + \phi_2^2)A_\mu A^\mu \\
&= (\partial_\mu\eta)(\partial^\mu\eta) + (\partial_\mu\xi)(\partial^\mu\xi) + \frac{2q}{\hbar c}\left[-(\partial_\mu\eta)\xi + (\partial_\mu\xi)\left(\eta + \frac{\mu}{\lambda}\right)\right]A^\mu \\
&\quad + \left(\frac{q}{\hbar c}\right)^2\left(\eta^2 + 2\eta\frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2} + \xi^2\right)A_\mu A^\mu
\end{aligned}$$

Putting it all together,

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) + \frac{q}{\hbar c}\left[-(\partial_\mu \eta)\xi + (\partial_\mu \xi)\eta\right] A^\mu \\
&\quad + \frac{\mu}{\lambda}\frac{q}{\hbar c}(\partial_\mu \xi)A^\mu + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2 \left(\eta^2 + 2\eta\frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2} + \xi^2\right) A_\mu A^\mu - \mu^2\eta^2 \\
&\quad - \frac{\lambda^2}{4}\left(\eta^4 + \xi^4 + 2\eta^2\xi^2\right) - \lambda\mu(\eta^3 + \eta\xi^2) + \frac{\mu^4}{4\lambda^2} - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} \\
&= \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi)\right] \\
&\quad + \left[-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2 \frac{\mu^2}{\lambda^2}A_\mu A^\mu\right] \\
&\quad + \frac{q}{\hbar c}\left[-(\partial_\mu \eta)\xi + (\partial_\mu \xi)\eta\right] A^\mu + \left(\frac{q}{\hbar c}\right)^2 \frac{\mu}{\lambda}\eta A_\mu A^\mu \\
&\quad + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2 \left(\eta^2 + \xi^2\right) A_\mu A^\mu - \frac{\lambda^2}{4}\left(\eta^4 + \xi^4 + 2\eta^2\xi^2\right) - \lambda\mu(\eta^3 + \eta\xi^2) \\
&\quad + \frac{\mu}{\lambda}\frac{q}{\hbar c}(\partial_\mu \xi)A^\mu + \frac{\mu^4}{4\lambda^2}. \checkmark
\end{aligned}$$

Problem 10.26



11
Neutrino Oscillations

Problem 11.1

To calculate the energy liberated in the creation of a sphere of radius R and uniform density ρ, imagine building it up by layers. When it is at radius r, a mass dm falls in (from infinity), giving up potential energy

$$dE = G\frac{m\,dm}{r} = \frac{G}{r}\left(\rho\frac{4}{3}\pi r^3\right)(\rho 4\pi r^2\,dr) = \frac{G}{3}(4\pi\rho)^2 r^4\,dr,$$

so the total energy of the sphere is

$$\begin{aligned} E &= \frac{GR^5}{15}(4\pi\rho)^2 = \frac{GR^5}{15}\left(\frac{3M}{R^3}\right)^2 = \frac{3}{5}\frac{GM^2}{R} \\ &= \frac{3}{5}\frac{(6.67\times10^{-11})(1.99\times10^{30})^2}{6.96\times10^8}\,\text{J} = 2.28\times10^{41}\,\text{J}. \end{aligned}$$

The solar luminosity is 3.85×10^{26} W (all numbers from the Particle Physics Booklet), so the lifetime is

$$\frac{2.28\times10^{41}}{3.85\times10^{26}}\,\text{s} = 5.91\times10^{14}\,\text{s} = \boxed{1.87\times10^7\,\text{yrs}}.$$

Problem 11.2

(a) Follow the derivation of Eq. 11.7, only this time assume the rest energy dominates:

$$E^2 = m^2c^4\left(1+\frac{|\mathbf{p}|^2}{m^2c^2}\right) \Rightarrow E \approx mc^2\left(1+\frac{1}{2}\frac{|\mathbf{p}|^2}{m^2c^2}\right) = mc^2 + \frac{|\mathbf{p}|^2}{2m}$$

$$\begin{aligned} E_2 - E_1 &= (m_2-m_1)c^2 + \frac{1}{2}|\mathbf{p}|^2\left(\frac{1}{m_2}-\frac{1}{m_1}\right) = (m_2-m_1)c^2 - T\left(\frac{m_2-m_1}{m}\right) \\ &\approx (m_2-m_1)c^2 \end{aligned}$$

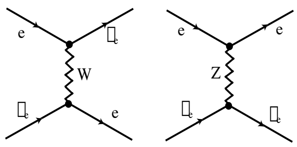
(where m is the average of the two masses, and T is the kinetic energy). From Eq. 11.6, the period of the oscillations is

$$\tau = \frac{2\pi\hbar}{E_2 - E_1} = \frac{2\pi\hbar}{(m_2 - m_1)c^2} = \frac{2\pi\, 6.58 \times 10^{-22}\,\text{MeV s}}{3.49 \times 10^{-12}\,\text{MeV}} = \boxed{1.18 \times 10^{-9}\,\text{s}}.$$

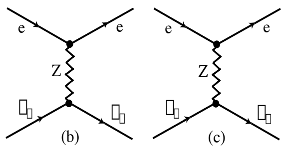
(b) The lifetime of K_S^0 is much shorter: 8.94×10^{-11} s; the lifetime of K_L^0 is much longer: 5.17×10^{-8} s. So a beam that starts out pure K^0 (say) will become pure K_L^0 long before it has a chance to oscillate to $\bar{K}^0$.

Problem 11.3

(**a**)

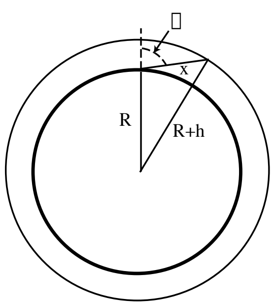


(**b** and **c**)



Problem 11.4

(**a**)



$(R+h)^2 = R^2 + x^2 - 2Rx\cos(\pi - \Theta) \Rightarrow R^2 + 2Rh + h^2 = R^2 + x^2 + 2Rx\cos\Theta;$

$$x^2 + (2R\cos\Theta)x - (2Rh + h^2) = 0 \Rightarrow \boxed{x = \sqrt{R^2\cos\Theta^2 + 2Rh + h^2} - R\cos\Theta}.$$

(We need the positive root, since $x > 0$.)

(**b**)

Problem 11.5

(**a**)

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \Rightarrow 1 - \left(\frac{v}{c}\right)^2 = \left(\frac{mc^2}{E}\right)^2 \Rightarrow v = c\sqrt{1 - (mc^2/E)^2}.$$

Ultrarelativistic $\Rightarrow E \gg mc^2$, so (keeping the first term in the binomial expansion)

$$v \approx c\left[1 - \frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right], \quad \frac{1}{v} \approx \frac{1}{c}\left[1 + \frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right].$$

(**b**) The time it takes a neutrino to reach earth is $t = L/v$, so the difference in arrival times is

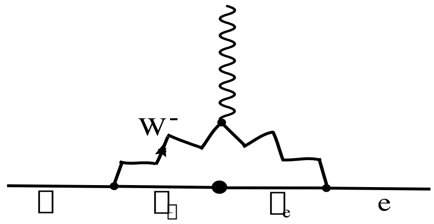
$$\begin{aligned}\tau &= t_2 - t_1 = L\left(\frac{1}{v_2} - \frac{1}{v_1}\right) \approx \frac{L}{c}\left[1 + \frac{1}{2}\left(\frac{mc^2}{E_2}\right)^2 - 1 - \frac{1}{2}\left(\frac{mc^2}{E_1}\right)^2\right] \\ &= \frac{L}{2c}(mc^2)^2\left(\frac{1}{E_2^2} - \frac{1}{E_1^2}\right) = \frac{L}{2c}(mc^2)^2\frac{(E_1 - E_2)(E_1 + E_2)}{E_1^2E_2^2}.\end{aligned}$$

$$\begin{aligned}mc^2 &= E_1E_2\sqrt{\frac{2c\tau}{L(E_1 - E_2)(E_1 + E_2)}} \\ &= (20)(30)\sqrt{\frac{2c(10)}{(1.7\times 10^5)c(365\cdot 24\cdot 60\cdot 60)(10)(50)}}\,\text{MeV} = \boxed{52\text{ eV}}.\end{aligned}$$

[Statistical analysis lowers this number a bit.]

Problem 11.6

(**a**)



(**b**)

$$\Delta E\,\Delta t \geq \frac{\hbar}{2} \Rightarrow \Delta t = \frac{\hbar}{2M_W c^2} = \frac{6.58 \times 10^{-22}}{2 \times 80.4 \times 10^3} = \boxed{4.09 \times 10^{-27}\,\mathrm{s}}.$$

$$d = (4.09 \times 10^{-27})(3 \times 10^8) = \boxed{1.23 \times 10^{-18}\,\mathrm{m}}.$$

In the time available, the neutrino travels *nowhere near* far enough for oscillations to occur. So this decay is still effectively forbidden.

12
What's Next

Problem 12.1

(a) Equation 10.132 gives the mass of the gauge particle under the Higgs mechanism:

$$m_A = 2\sqrt{\pi}\left(\frac{q\mu}{\lambda c^2}\right) \quad \Rightarrow \quad M_W = 2\sqrt{\pi}\frac{v}{c^2}\frac{g_w}{2}\sqrt{\frac{\hbar c}{4\pi}} = \boxed{\frac{g_w v}{2c^2}\sqrt{\hbar c}}.$$

Thus

$$\sqrt{\hbar c}\, v = \frac{2M_W c^2}{g_w}. \quad \checkmark$$

(b) From Problem 10.21, the Yukawa coupling $\mathcal{L}_{\text{int}} = -\alpha_Y \bar{\psi}\psi\phi$ yields a naive vertex factor $i\alpha_Y$. When the scalar field ϕ shifts, according to Eq. 10.130, $\phi = \eta + \mu/\lambda = \eta + v$, the Yukawa coupling generates a mass term for the quark or lepton field ψ:

$$-\alpha_Y \bar{\psi}\psi v = -mc^2\bar{\psi}\psi \quad \Rightarrow \quad \alpha_Y = \frac{mc^2}{v}$$

plus a residual Yukawa coupling whose vertex factor is evidently $\boxed{imc^2/v}$. [As I mentioned before Eq. 10.101, this procedure doesn't get the constants right; the correct vertex factor is $-imc^2/(v\sqrt{\hbar c})$.]

(c) The relevant terms in Eq. 10.136 are

$$i\mathcal{L}_{\text{int}} = i\left[\frac{\mu}{\lambda}\left(\frac{q}{\hbar c}\right)^2 g^{\mu\nu}\eta A_\mu A_\nu - \lambda\mu\eta^3\right].$$

The first term gives the hWW and hZZ couplings, with the vertex factor (see remark before Eq. 10.101)

$$i\frac{4\pi}{\hbar c}\frac{\mu}{\lambda}\left(\frac{q}{\hbar c}\right)^2 g^{\mu\nu} = \boxed{i\frac{(M_m c^2)^2}{v(\hbar c)^3}g^{\mu\nu}}$$

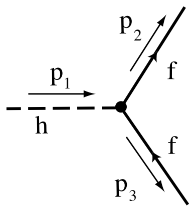
(where M_m stands for the mass of the W or the Z, as the case may be). [The correct vertex factor is $i(2M^2c^3/v\sqrt{\hbar c})g^{\mu\nu}$.]

The second term gives the hhh coupling, with vertex factor$-i\lambda\mu = -i\mu^2/v$. But (comparing Eqs. 10.11 and 10.136) the mass of the Higgs is given by $\mu^2 = \frac{1}{2}(m_h c/\hbar)^2$, so the hhh vertex factor is

$$\boxed{-i\frac{1}{2v}\left(\frac{m_h c}{\hbar}\right)^2}.$$

[Actually, $-3im_h^2c^3/v$.]

Problem 12.2



(a) The vertex factor (Problem 12.1) is $imc^2/v\sqrt{\hbar c}$, where m is the mass of the quark or lepton, so

$$\mathcal{M} = i\left[\bar{v}(3)\left(\frac{imc^2}{v\sqrt{\hbar c}}\right)u(2)\right] = -\frac{mc^2}{v\sqrt{\hbar c}}\left[\bar{v}(3)u(2)\right].$$

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \left(\frac{mc^2}{v\sqrt{\hbar c}}\right)^2 \mathrm{Tr}\left[(\not{p}_2 + mc)(\not{p}_3 - mc)\right] \\ &= \left(\frac{mc^2}{v\sqrt{\hbar c}}\right)^2\left[\mathrm{Tr}(\not{p}_2\not{p}_3 - (mc)^2\mathrm{Tr}(1)\right] = 4\left(\frac{mc^2}{v\sqrt{\hbar c}}\right)^2\left[p_2\cdot p_3 - (mc)^2\right].\end{aligned}$$

In the CM,

$$p_2 = \left(\frac{E}{c}, \mathbf{p}\right),\ p_3 = \left(\frac{E}{c}, -\mathbf{p}\right),\ \mathbf{p}^2 = \left(\frac{E}{c}\right)^2 - (mc)^2,\ p_2\cdot p_3 = \left(\frac{E}{c}\right)^2 + \mathbf{p}^2,$$

so

$$\langle|\mathcal{M}|^2\rangle = 4\left(\frac{mc^2}{v\sqrt{\hbar c}}\right)^2 2\left[\left(\frac{E}{c}\right)^2 - (mc)^2\right].$$

Conservation of energy says $E = (m_h c^2/2)$, so

$$\langle|\mathcal{M}|^2\rangle = 8\left(\frac{mc^2}{v\sqrt{\hbar c}}\right)^2\left[\left(\frac{m_h c}{2}\right)^2 - (mc)^2\right] = 2\left(\frac{mc^3}{v\sqrt{\hbar c}}\right)^2\left(m_h^2 - 4m^2\right).$$

The decay rate is (Eq. 6.35 with $|\mathbf{p}| = \sqrt{(E/c)^2 - (mc)^2} = (c/2)\sqrt{m_h^2 - 4m^2}$):

$$\Gamma = \frac{\frac{c}{2}\sqrt{m_h^2 - 4m^2}}{8\pi\hbar m_h^2 c} 2 \left(\frac{mc^3}{v\sqrt{\hbar c}}\right)^2 \left(m_h^2 - 4m^2\right) = \frac{1}{8\pi\hbar m_h^2}\left(\frac{mc^3}{v\sqrt{\hbar c}}\right)^2 \left(m_h^2 - 4m^2\right)^{3/2}$$

Or, using Eq. 12.1 to express v in terms of M_W:

$$\Gamma = \frac{1}{8\pi\hbar m_h^2}\left(\frac{mc^3 g_w}{2M_W c^2}\right)^2 \left(m_h^2 - 4m^2\right)^{3/2} = \frac{m^2c^2g_w^2}{32\pi\hbar m_h^2 M_W^2}\left(m_h^2 - 4m^2\right)^{3/2}.$$

Finally, writing $g_w = \sqrt{4\pi\alpha_w}$:

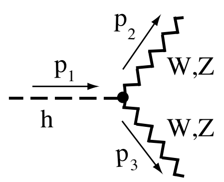
$$\Gamma = \frac{\alpha_w}{8\hbar} m_h c^2 \left(\frac{m}{M_W}\right)^2 \left[1 - \left(\frac{2m}{m_h}\right)^2\right]^{3/2}.$$

(b)

$$\begin{aligned}\frac{\Gamma(b\bar{b})}{\Gamma(c\bar{c})} &= \left(\frac{m_b}{m_c}\right)^2 \left[\frac{1-(2m_b/m_h)^2}{1-(2m_c/m_h)^2}\right]^{3/2} \\ &= \left(\frac{4.3}{1.2}\right)^2 \left[\frac{1-(8.6/120)^2}{1-(2.4/120)^2}\right]^{3/2} = (12.8)(0.9953)^{3/2} = \boxed{13}.\end{aligned}$$

$$\begin{aligned}\frac{\Gamma(b\bar{b})}{\Gamma(\tau^+\tau^-)} &= 3\left(\frac{m_b}{m_\tau}\right)^2 \left[\frac{1-(2m_b/m_h)^2}{1-(2m_\tau/m_h)^2}\right]^{3/2} \\ &= 3\left(\frac{4.3}{1.78}\right)^2 \left[\frac{1-(8.6/120)^2}{1-(3.55/120)^2}\right]^{3/2} = 3(5.84)(0.9958)^{3/2} = \boxed{17}.\end{aligned}$$

Problem 12.3



The vertex factor is (Problem 12.1):

$$2i\frac{M^2c^3}{v\sqrt{\hbar c}}g^{\mu\nu},$$

where M is the mass of the W or the Z, as the case may be. The amplitude is

$$\mathcal{M} = i\epsilon_\mu^*(2)\left[i\frac{2M^2c^3}{v\sqrt{\hbar c}}g^{\mu\nu}\right]\epsilon_\nu^*(3) = -\frac{2M^2c^3}{v\sqrt{\hbar c}}\left[\epsilon_\mu^*(2)\epsilon^{*\mu}(3)\right].$$

$$|\mathcal{M}|^2 = \left(\frac{2M^2c^3}{v\sqrt{\hbar c}}\right)^2\left[\epsilon_\mu^*(2)\epsilon^{*\mu}(3)\right]\left[\epsilon_\nu(2)\epsilon^\nu(3)\right].$$

Summing over the outgoing spins, using Eq. 9.158:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \left(\frac{2M^2c^3}{v\sqrt{\hbar c}}\right)^2\left[-g_{\mu\nu} + \frac{p_{2\mu}p_{2\nu}}{(Mc)^2}\right]\left[-g^{\mu\nu} + \frac{p_3^\mu p_3^\nu}{(Mc)^2}\right]\\ &= \left(\frac{2M^2c^3}{v\sqrt{\hbar c}}\right)^2\left[4 - \frac{p_2\cdot p_2}{(Mc)^2} - \frac{p_3\cdot p_3}{(Mc)^2} + \frac{(p_2\cdot p_3)^2}{(Mc)^4}\right]\\ &= \left(\frac{2M^2c^3}{v\sqrt{\hbar c}}\right)^2\left[2 + \frac{(p_2\cdot p_3)^2}{(Mc)^4}\right] = \left(\frac{2c}{v\sqrt{\hbar c}}\right)^2\left[2(Mc)^4 + (p_2\cdot p_3)^2\right].\end{aligned}$$

In the CM frame

$$p_2 = \left(\frac{E}{c},\mathbf{p}\right), \quad p_3 = \left(\frac{E}{c},-\mathbf{p}\right), \quad \mathbf{p}^2 = \left(\frac{E}{c}\right)^2 - (Mc)^2, \quad E = \frac{1}{2}m_hc^2$$

$$p_2\cdot p_3 = \left(\frac{E}{c}\right)^2 + \mathbf{p}^2 = 2\left(\frac{E}{c}\right)^2 - (Mc)^2 = \frac{c^2}{2}(m_h^2 - 2M^2),$$

$$\begin{aligned}\left[2(Mc)^4 + (p_2\cdot p_3)^2\right] &= 2(Mc)^4 + \frac{c^4}{4}\left(m_h^4 - 4m_h^2M^2 + 4M^4\right)\\ &= \frac{c^4}{4}\left(m_h^4 - 4m_h^2M^2 + 12M^4\right),\end{aligned}$$

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{c^3}{v\sqrt{\hbar c}}\right)^2\left(m_h^4 - 4m_h^2M^2 + 12M^4\right).$$

Put this into Eq. 6.35, with $|\mathbf{p}| = \sqrt{(m_hc/2)^2 - (Mc)^2} = (c/2)\sqrt{m_h^2 - 4M^2}$,

$$\begin{aligned}\Gamma &= \frac{S(c/2)\sqrt{m_h^2 - 4M^2}}{8\pi\hbar m_h^2c}\left(\frac{c^3}{v\sqrt{\hbar c}}\right)^2\left(m_h^4 - 4m_h^2M^2 + 12M^4\right)\\ &= S\frac{m_h^3c^5}{16\pi\hbar^2v^2}\left[1 - 4\left(\frac{M}{m_h}\right)^2 + 12\left(\frac{M}{m_h}\right)^4\right]\sqrt{1 - \left(\frac{2M}{m_h}\right)^2}.\end{aligned}$$

Or, using Eq. 12.1 with $g_w = \sqrt{4\pi\alpha_w}$:

$$\Gamma = \boxed{S\frac{\alpha_w m_h c^2}{16\hbar}\left(\frac{m_h}{M_W}\right)^2\left[1 - 4\left(\frac{M}{m_h}\right)^2 + 12\left(\frac{M}{m_h}\right)^4\right]\sqrt{1 - \left(\frac{2M}{m_h}\right)^2}}.$$

The statistical factor S is 1 for $h \to W^+ + W^-$, and $1/2$ for $h \to Z + Z$ (because in this case there are two identical particles in the final state).
(b)

$$\frac{\Gamma(WW)}{\Gamma(ZZ)} = 2\frac{\left[1 - 4\left(\frac{M_W}{m_h}\right)^2 + 12\left(\frac{M_W}{m_h}\right)^4\right]\sqrt{1 - \left(\frac{2M_W}{m_h}\right)^2}}{\left[1 - 4\left(\frac{M_Z}{m_h}\right)^2 + 12\left(\frac{M_Z}{m_h}\right)^4\right]\sqrt{1 - \left(\frac{2M_Z}{m_h}\right)^2}}.$$

With $M_W = 80$, $M_Z = 91$, and $m_h = 200$,

$$\frac{\Gamma(WW)}{\Gamma(ZZ)} = \boxed{2.8}.$$

Problem 12.4

The Super-K tank holds 50,000 tons, or

$$(5 \times 10^4 \text{ tons})(2000 \text{ lbs/ton})((4.45 \text{ N/lb})\frac{1}{9.8 \text{ m/s}^2} = 4.5 \times 10^7 \text{ kg}.$$

Roughly half of the mass is neutrons, but let's imagine it is *entirely* protons; then the number of protons in the tank is

$$\frac{4.5 \times 10^7 \text{ kg}}{1.67 \times 10^{-27} \text{ kg}} = 2.7 \times 10^{34} \text{ protons}.$$

Suppose the experiment is to run for one year, and in that time we hope to see 5 proton decays. The decay rate is $1/\tau$, so

$$5 = \frac{1}{\tau}(2.7 \times 10^{34})(1 \text{ yr}) \Rightarrow \tau = \frac{2.7 \times 10^{34}}{5} \text{ yr} = \boxed{5 \times 10^{33} \text{ yr}}.$$

The current limit is 10^{33} years, so we are very close to the practical limit on such experiments.

Problem 12.5

- Decay of the muon (Eq. 9.34): $\Gamma \sim \dfrac{1}{12(8\pi)^3}\dfrac{g_w^4}{\hbar}\dfrac{(m_\mu c^2)^5}{(M_W c^2)^4}.$
- Decay of the neutron (Eq. 9.60): $\Gamma \sim \dfrac{1}{4\pi^3 2^4}\dfrac{g_w^4}{\hbar}\dfrac{(m_e c^2)^5}{(M_W c^2)^4}\left(\dfrac{m_n - m_p}{m_e}\right)^5.$

- Semileptonic decays (Ex. 9.3): $\Gamma \sim \frac{1}{30\pi^3 2^4} \frac{g_w^4}{\hbar} \frac{(\Delta mc^2)^5}{(M_W c^2)^4}$.

The pion decay formula (Eq. 9.76) is not so helpful here, since we don't know f_π, but apart from numerical factors (including g_w), the generic formula seems to be

$$\boxed{\tau \sim \hbar \frac{(Mc^2)^4}{(mc^2)^5}},$$

where m is the mass of the decaying particle and M is the mass of the mediator responsible. For proton decay, $mc^2 \approx 1$ GeV, $Mc^2 \approx 10^{16}$ GeV, so

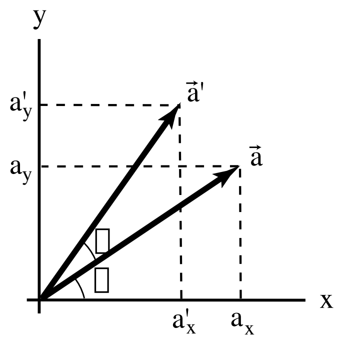
$$\tau \sim 10^{-24} \frac{(10^{16})^4}{(1)^5} = 10^{40} \text{ s},$$

or, since there are $(365)(24)(60)(60) = 3 \times 10^7$ seconds in a year, $\boxed{\tau \sim 10^{32} \text{ yr}}$. Obviously, tweaking the numerical factors could change this by several orders of magnitude, but essential point remains: this is an *extremely* long lifetime.

Problem 12.6

$$\begin{aligned}
&|\mathcal{M}|^2 - |\tilde{\mathcal{M}}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2 - |\tilde{\mathcal{M}}_1 + \tilde{\mathcal{M}}_2|^2 \\
&= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathcal{M}_1\mathcal{M}_2^* + \mathcal{M}_1^*\mathcal{M}_2 - |\tilde{\mathcal{M}}_1|^2 - |\tilde{\mathcal{M}}_2|^2 - \tilde{\mathcal{M}}_1\tilde{\mathcal{M}}_2^* - \tilde{\mathcal{M}}_1^*\tilde{\mathcal{M}}_2 \\
&= |\mathcal{M}_1|e^{i\phi_1}e^{i\theta_1}|\mathcal{M}_2|e^{-i\phi_2}e^{-i\theta_2} + |\mathcal{M}_1|e^{-i\phi_1}e^{-i\theta_1}|\mathcal{M}_2|e^{i\phi_2}e^{i\theta_2} \\
&\quad -|\mathcal{M}_1|e^{i\phi_1}e^{-i\theta_1}|\mathcal{M}_2|e^{-i\phi_2}e^{i\theta_2} - |\mathcal{M}_1|e^{-i\phi_1}e^{i\theta_1}|\mathcal{M}_2|e^{i\phi_2}e^{-i\theta_2} \\
&= |\mathcal{M}_1||\mathcal{M}_2| \left\{ e^{i(\theta_1-\theta_2)} \left[e^{i(\phi_1-\phi_2)} - e^{-i(\phi_1-\phi_2)} \right] + e^{-i(\theta_1-\theta_2)} \left[e^{-i(\phi_1-\phi_2)} - e^{i(\phi_1-\phi_2)} \right] \right\} \\
&= 2i|\mathcal{M}_1||\mathcal{M}_2| \sin(\phi_1 - \phi_2) \left[e^{i(\theta_1-\theta_2)} - e^{-i(\theta_1-\theta_2)} \right] \\
&= -4|\mathcal{M}_1||\mathcal{M}_2| \sin(\phi_1 - \phi_2) \sin(\theta_1 - \theta_2). \ \text{qed}
\end{aligned}$$

Problem 12.7



(a)

$$a_x = a\cos\phi, \qquad a_y = a\sin\phi$$

$$\begin{aligned} a'_x &= a\cos(\theta+\phi) = a\left[\cos\theta\cos\phi - \sin\theta\sin\phi\right] = \cos\theta\, a_x - \sin\theta\, a_y, \quad \checkmark \\ a'_y &= a\sin(\theta+\phi) = a\left[\sin\theta\cos\phi + \cos\theta\sin\phi\right] = \sin\theta\, a_x + \cos\theta\, a_y, \quad \checkmark \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{a}'\cdot\mathbf{b}' &= a'_x b'_x + a'_y b'_y = (\cos\theta\, a_x - \sin\theta\, a_y)(\cos\theta\, b_x - \sin\theta\, b_y) \\ &\qquad + (\cos\theta\, a_y + \sin\theta\, a_x)(\cos\theta\, b_y + \sin\theta\, b_x) \\ &= a_x b_x(\cos^2\theta + \sin^2\theta) + a_x b_y(-\cos\theta\sin\theta + \sin\theta\cos\theta) \\ &\qquad + a_y b_x(-\cos\theta\sin\theta + \sin\theta\cos\theta) + a_y b_y(\cos^2\theta + \sin^2\theta) \\ &= a_x b_x + a_y b_y. \quad \checkmark \end{aligned}$$

(c)

$$\sin(d\theta) \approx d\theta, \quad \cos(d\theta) \approx 1, \quad \boxed{a'_x = a_x - (d\theta)a_y, \;\; a'_y = a_y + (d\theta)a_x}.$$

(d)

$$\begin{aligned} a'_x b'_x + a'_y b'_y &= \left[a_x - (d\theta)a_y\right]\left[b_x - (d\theta)b_y\right] + \left[a_y + (d\theta)a_x\right]\left[b_y + (d\theta)b_x\right] \\ &= a_x b_x\left[1 + (d\theta)^2\right] + a_x b_y\left[-d\theta + d\theta\right] + a_y b_x\left[d\theta - d\theta\right] \\ &\quad + a_y b_y\left[1 + (d\theta)^2\right] = (a_x b_x + a_y b_y)\left[1 + (d\theta)^2\right] = (a_x b_x + a_y b_y) \quad \checkmark \end{aligned}$$

(to first order in $d\theta$).

Problem 12.8

(a)

$$\delta\phi^* = (\delta\phi)^* = (2\epsilon^\dagger\gamma^0\psi)^\dagger = 2\psi^\dagger(\gamma^0)^\dagger\epsilon = 2\psi^\dagger\gamma^0\epsilon = 2\bar{\psi}\epsilon$$

(recall that ${\gamma^0}^\dagger = \gamma^0$). Similarly

$$\begin{aligned}\delta\bar{\psi} &= \delta(\psi^\dagger\gamma^0) = (\delta\psi)^\dagger\gamma^0 = (i/\hbar c)(\gamma^\mu\epsilon\partial_\mu\phi)^\dagger\gamma^0 \\ &= (i/\hbar c)\epsilon^\dagger{\gamma^\mu}^\dagger\gamma^0\partial_\mu\phi^* = (i/\hbar c)(\epsilon^\dagger\gamma^0)\gamma^0{\gamma^\mu}^\dagger\gamma^0\partial_\mu\phi^* = (i/\hbar c)\bar{\epsilon}\gamma^\mu\partial_\mu\phi^*\end{aligned}$$

(recall that $\gamma^0\gamma^0 = 1$ and $\gamma^0{\gamma^\mu}^\dagger\gamma^0 = \gamma^\mu$—see Problem 7.15).

(b)

$$\begin{aligned}\delta\mathscr{L}_1 &= \frac{1}{2}\left[\partial^\mu(\delta\phi^*)(\partial_\mu\phi) + (\partial^\mu\phi^*)\partial_\mu(\delta\phi)\right] = \partial^\mu(\bar{\psi}\epsilon)(\partial_\mu\phi) + (\partial^\mu\phi^*)\partial_\mu(\bar{\epsilon}\psi) \\ &= (\partial^\mu\phi)(\partial_\mu\bar{\psi})\epsilon + \bar{\epsilon}(\partial^\mu\phi^*)(\partial_\mu\psi).\end{aligned}$$

(c)

$$\begin{aligned}\delta\mathscr{L}_2 &= i\hbar c\left[(\delta\bar{\psi})\gamma^\mu(\partial_\mu\psi) + \bar{\psi}\gamma^\mu\partial_\mu(\delta\psi)\right] = -(\bar{\epsilon}\gamma^\nu\partial_\nu\phi^*)\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu(\gamma^\nu\epsilon\partial_\nu\phi) \\ &= -\bar{\epsilon}(\partial_\nu\phi^*)\gamma^\nu\gamma^\mu(\partial_\mu\psi) + \bar{\psi}\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu\phi)\epsilon.\end{aligned}$$

Now, $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu$ can be written more symmetrically as $(1/2)(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial^2$, the d'Alembertian. So

$$\delta\mathscr{L}_2 = -\bar{\epsilon}(\partial_\nu\phi^*)\gamma^\nu\gamma^\mu(\partial_\mu\psi) + \partial_\mu[\bar{\psi}(\partial^\mu\phi)\epsilon] - (\partial^\mu\phi)(\partial_\mu\bar{\psi})\epsilon.$$

The third term on the right is $-\delta\mathscr{L}_1 + \bar{\epsilon}(\partial^\mu\phi^*)(\partial_\mu\psi)$, so

$$\delta\mathscr{L}_2 = -\delta\mathscr{L}_1 + \partial_\mu[\bar{\psi}(\partial^\mu\phi)\epsilon] + \bar{\epsilon}\left\{(\partial^\mu\phi^*)(\partial_\mu\psi) - (\partial_\nu\phi^*)\gamma^\nu\gamma^\mu(\partial_\mu\psi)\right\}.$$

The term in curly brackets can be written as

$$(\partial_\nu\phi^*)\left[g^{\mu\nu} - \gamma^\nu\gamma^\mu\right](\partial_\mu\psi) = -i\sigma^{\mu\nu}(\partial_\nu\phi^*)(\partial_\mu\psi)$$

(I used the anticommutation relation for the Dirac matrices, Eq. 7.15, and the definition of $\sigma^{\mu\nu}$, Eq. 7.69). But

$$\begin{aligned}&\partial_\mu\left\{\sigma^{\mu\nu}\left[\phi^*(\partial_\nu\psi) - (\partial_\nu\phi^*)\psi\right]\right\} \\ &= \sigma^{\mu\nu}\left[(\partial_\mu\phi^*)(\partial_\nu\psi) + \phi^*(\partial_\mu\partial_\nu\psi) - (\partial_\mu\partial_\nu\phi^*)\psi - (\partial_\nu\phi^*)(\partial_\mu\psi)\right] \\ &= -2\sigma^{\mu\nu}(\partial_\nu\phi^*)(\partial_\mu\psi)\end{aligned}$$

(since $\sigma^{\mu\nu}$ is antisymmetric, the contraction with $\partial_\mu\partial_\nu$ is zero, and $\sigma^{\mu\nu}t_{\mu\nu} = -\sigma^{\mu\nu}t_{\nu\mu}$ for any tensor $t_{\mu\nu}$). Putting this together, we have

$$\begin{aligned}\delta\mathscr{L}_2 &= -\delta\mathscr{L}_1 + \partial_\mu[\bar{\psi}(\partial^\mu\phi)\epsilon] + \frac{i}{2}\bar{\epsilon}\partial_\mu\left\{\sigma^{\mu\nu}\left[\phi^*(\partial_\nu\psi) - (\partial_\nu\phi^*)\psi\right]\right\} \\ &= -\delta\mathscr{L}_1 + \partial_\mu Q^\mu.\end{aligned}$$

(d)

$$\delta\mathscr{L}_3 = -\frac{1}{2}\left[(\delta\phi^*)\phi + \phi^*(\delta\phi)\right] = -\left(\frac{mc}{\hbar}\right)^2(\bar{\psi}\epsilon\phi + \phi^*\bar{\epsilon}\psi).$$

$$\delta\mathscr{L}_4 = -mc^2\left[(\delta\bar{\psi})\psi + \bar{\psi}(\delta\psi)\right] = -i\left(\frac{mc}{\hbar}\right)\left[\bar{\epsilon}\gamma^\mu(\partial_\mu\phi^*)\psi - \bar{\psi}\gamma^\mu(\partial_\mu\phi)\epsilon\right].$$

(e) Using Eq. 10.15 (and its adjoint) to replace the derivatives of ψ (and $\bar{\psi}$),

$$\begin{aligned}\partial_\mu R^\mu &= i\left(\frac{mc}{\hbar}\right)\left[-\bar{\epsilon}\gamma^\mu(\partial_\mu\phi^*)\psi - \bar{\epsilon}\gamma^\mu\phi^*(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\epsilon\phi + \bar{\psi}\gamma^\mu\epsilon(\partial_\mu\phi)\right]\\ &= i\left(\frac{mc}{\hbar}\right)\left[-\bar{\epsilon}\gamma^\mu(\partial_\mu\phi^*)\psi + \bar{\psi}\gamma^\mu\epsilon(\partial_\mu\phi)\right] - \left(\frac{mc}{\hbar}\right)^2\left[\bar{\epsilon}\phi^*\psi + \bar{\psi}\epsilon\phi\right]\\ &= \delta\mathscr{L}_3 + \delta\mathscr{L}_4.\end{aligned}$$

Problem 12.9

(a)

$c = 3.00\times10^8\,\text{m/s};\ \hbar = 1.055\times10^{-34}\,\text{kg}\,\text{m}^2/\text{s};\ G = 6.67\times10^{-11}\text{m}^3/(\text{kg}\,\text{s}^2).$

The dimensions of $c^p\hbar^q G^r$ are

$$\left(\frac{\text{m}}{\text{s}}\right)^p\left(\frac{\text{kg}\,\text{m}^2}{\text{s}}\right)^q\left(\frac{\text{m}^3}{\text{kg}\,\text{s}^2}\right)^r = \text{m}^{p+2q+3r}\text{s}^{-p-q-2r}\text{kg}^{q-r},$$

so for l_P we want

$$q = r;\quad -p-q-2r = 0 \Rightarrow p = -3r;\quad p+2q+3r = 1 \Rightarrow q = 1/2.$$

Evidently

$$l_P = c^{-3/2}\hbar^{1/2}G^{1/2} = \boxed{\sqrt{\frac{\hbar G}{c^3}} = 1.6\times10^{-35}\,\text{m}}.$$

For t_P we want

$$q = r;\quad p+2q+3r = 0 \Rightarrow p = -5r;\quad -p-q-2r = 1 \Rightarrow r = 1/2.$$

Evidently

$$t_P = c^{-5/2}\hbar^{1/2}G^{1/2} = \boxed{\sqrt{\frac{\hbar G}{c^5}} = 5.38\times10^{-44}\,\text{s}}.$$

For m_P we want

$$q-r = 1 \Rightarrow q = r+1;\quad p+2q+3r = 0 \Rightarrow p = -5r-2;$$

$$-p-q-2r = 0 \Rightarrow 2r+1 = 0 \Rightarrow r = -1/2.$$

Evidently

$$m_P = c^{1/2}\hbar^{1/2}G^{-1/2} = \boxed{\sqrt{\frac{c\hbar}{G}} = 2.18 \times 10^{-8}\,\text{kg}}.$$

$$E_P = m_P c^2 = 1.96 \times 10^9\ \text{J} = \frac{1.96 \times 10^9}{1.60 \times 10^{-19}}\,\text{eV} = \boxed{1.23 \times 10^{19}\,\text{GeV}}.$$

(b) Coulomb ($F = q_1 q_2/r^2$) $\rightarrow$ Newton ($F = G m_1 m_2/r^2$) $\Rightarrow e^2 \rightarrow Gm^2$, so

$$\boxed{\alpha_G = \sqrt{\frac{Gm^2}{\hbar c}}} = \begin{cases} \sqrt{\frac{(6.67\times10^{-11})(9.11\times10^{-31})^2}{(1.055\times10^{-34})(3.00\times10^{8})}} = \boxed{4.18 \times 10^{-23}} & \text{(i)} \\ \sqrt{\frac{G}{\hbar c}\frac{c\hbar}{G}} = \boxed{1} & \text{(ii)} \end{cases}$$

Problem 12.10

$$\frac{GMm}{r^2} = m\frac{v^2}{r} \Rightarrow \boxed{v = \sqrt{\frac{GM}{r}}}.$$

Problem 12.11

Escape velocity:

$$\frac{1}{2}mv^2 - G\frac{Mm}{R} = 0 \Rightarrow v^2 = \frac{2G}{R}\frac{4}{3}\pi R^3 \rho = \frac{8\pi}{3}GR^2\rho = H^2R^2.$$

So

$$\rho = \frac{3H^2}{8\pi G}.$$

The Particle Physics Booklet gives $H = 7.5 \times 10^{-11}$ /yr $= 2.4 \times 10^{-18}$ /s, so

$$\rho = \frac{3(2.4 \times 10^{-18})^2}{8\pi(6.67 \times 10^{-11})} = \boxed{1.0 \times 10^{-26}\,\text{kg/m}^3}.$$

(For comparison, the density of water is 1000 kg/m^3.)

A
The Dirac Delta Function

Problem A.1

(a)

$$2(1^2) + 7(1) + 3 = \boxed{12}.$$

(b)

$\boxed{\text{Zero}}$ (π is outside the domain of integration.)

Problem A.2

$$g(x) = \sqrt{x^2+1} - x - 1; \quad g'(x) = \frac{1}{2}\frac{1}{\sqrt{x^2+1}}2x - 1 = \frac{x}{\sqrt{x^2+1}} - 1$$

$$\text{Zero of } g: \ \sqrt{x^2+1} = x + 1 \Rightarrow x^2 + 1 = x^2 + 2x + 1 \Rightarrow x = 0;$$

$$g'(0) = -1; \quad |g'(0)| = 1, \quad \text{so } \boxed{\delta(\sqrt{x^2+1} - x - 1) = \delta(x)}.$$

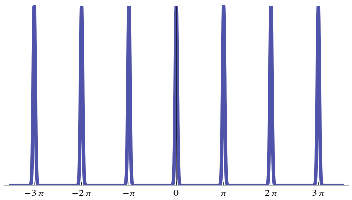
Problem A.3

$$g(x) = \sin x \Longrightarrow g'(x) = \cos x$$

$$g(x) = 0 \Longrightarrow \sin x = 0 \Longrightarrow x = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

$$g'(n\pi) = \cos n\pi = (-1)^n \Longrightarrow |g'(n\pi)| = 1.$$

$$\boxed{\delta(\sin x) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi)}.$$

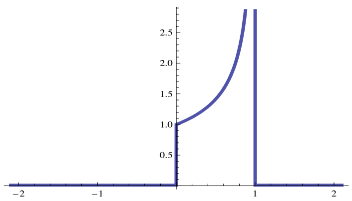


Problem A.4

$$g(x) \equiv x^2 - 2x + y = 0 \Rightarrow x_\pm = 1 \pm \sqrt{1-y}.$$
$$g'(x) = 2x - 2; \quad g'(x_\pm) = 2(x_\pm - 1) = \pm\sqrt{1-y}; \quad |g'(x_\pm)| = 2\sqrt{1-y}.$$
$$\delta(g(x)) = \frac{1}{2\sqrt{1-y}}\left[\delta(x-x_+) + \delta(x-x_-)\right]$$

$0 < x_\pm < 2 \Leftrightarrow 0 < 1 \pm \sqrt{1-y} < 2$, or $-1 < \pm\sqrt{1-y} < 1$. But if a is in the range $-1 < a < 1$, so too is $-a$, so $x_\pm$ are both in the domain of the integral provided $\sqrt{1-y} < 1$, which is to say $0 < y < 1$. Conclusion:

$$f(y) \equiv \int_0^2 \delta(g(x))\,dx = \boxed{\begin{cases} 1/\sqrt{1-y}, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}}$$



Problem A.5

$$\int_{-1}^{5} x^4 \frac{d^2}{dx^2}\left[\delta(x-3)\right]\,dx = x^4\,\frac{d}{dx}\delta(x-3)\Big|_{-1}^{5} - \int_{-1}^{5}(4x^3)\,\frac{d}{dx}\delta(x-3)\,dx$$

Here the first term in the right-hand side is zero because $\delta(x-3)$ has slope zero at $x = 5$ and at $x = -1$. The remaining integral is

$$-4x^3\,\delta(x-3)|_{-1}^{5} + \int_{-1}^{5} 12x^2\delta(x-3)dx = 12(3)^2 = \boxed{108},$$

The first term is zero, because $\delta(x-3)$ is zero at $x = 5$ and at $x = -1$.

Problem A.6

$\theta(2x-4) = 0$, if $x < 2$, and 1, if $x > 2$, so

$$\int_{-1}^{5} \theta(2x-4)e^{-3x}\,dx = \int_{2}^{5} e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big|_{2}^{5} = -\frac{1}{3}\left(e^{-15}-e^{-6}\right)$$
$$= \boxed{0.00082615}.$$

Problem A.7

$$\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) = (3,2,1)\cdot(-2,0,2) = -6+0+2 = \boxed{-4},$$

provided the point $\mathbf{b}$ lies within the domain of integration. Distance from center $(2,2,2)$ to $\mathbf{b}$ is

$$d = |(3,2,1)-(2,2,2)| = |(1,0,-1)| = \sqrt{1^2+0^2+1^2} = \sqrt{2} = 1.414 < 1.5,$$

so $\mathbf{b}$ *is* within the sphere.
