Solutions to Problems in Goldstein, *Classical Mechanics*, Second Edition

Homer Reid

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Chapter 3

Problem 3.1

A particle of mass m is constrained to move under gravity without friction on the inside of a paraboloid of revolution whose axis is vertical. Find the one-dimensional problem equivalent to its motion. What is the condition on the particle's initial velocity to produce circular motion? Find the period of small oscillations about this circular motion.

We'll take the paraboloid to be defined by the equation $z = \alpha r^2$. The kinetic and potential energies of the particle are

$$\begin{aligned} T &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + 4\alpha^2 r^2 \dot{r}^2) \\ V &= mgz = mg\alpha r^2. \end{aligned}$$

Hence the Lagrangian is

$$L = \frac{m}{2}\left[(1 + 4\alpha^2 r^2)\dot{r}^2 + r^2\dot{\theta}^2\right] - mg\alpha r^2.$$

This is cyclic in θ, so the angular momentum is conserved:

$$l = mr^2\dot{\theta} = \text{constant}.$$

For r we have the derivatives

$$\begin{aligned}
\frac{\partial L}{\partial r} &= 4\alpha^2 m r \dot{r}^2 + m r \dot{\theta}^2 - 2mg\alpha r \\
\frac{\partial L}{\partial \dot{r}} &= m(1 + 4\alpha^2 r^2)\dot{r} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 8m\alpha^2 r \dot{r}^2 + m(1 + 4\alpha^2 r^2)\ddot{r}.
\end{aligned}$$

Hence the equation of motion for r is

$$8m\alpha^2 r \dot{r}^2 + m(1 + 4\alpha^2 r^2)\ddot{r} = 4\alpha^2 m r \dot{r}^2 + m r \dot{\theta}^2 - 2mg\alpha r$$

or

$$m(1 + 4\alpha^2 r^2)\ddot{r} + 4m\alpha^2 r \dot{r}^2 - m r \dot{\theta}^2 + 2mg\alpha r = 0.$$

In terms of the constant angular momentum, we may rewrite this as

$$m(1 + 4\alpha^2 r^2)\ddot{r} + 4m\alpha^2 r \dot{r}^2 - \frac{l^2}{mr^3} + 2mg\alpha r = 0.$$

So this is the differential equation that determines the time evolution of r.

If initially $\dot{r} = 0$, then we have

$$m(1 + 4\alpha^2 r^2)\ddot{r} + -\frac{l^2}{mr^3} + 2mg\alpha r = 0.$$

Evidently, $\ddot{r}$ will then vanish—and hence $\dot{r}$ will remain 0, giving circular motion—if

$$\frac{l^2}{mr^3} = 2mg\alpha r$$

or

$$\dot{\theta} = \sqrt{2g\alpha}.$$

So if this condition is satisfied, the particle will execute circular motion (assuming its initial r velocity was zero). It's interesting to note that the condition on $\dot{\theta}$ for circular motion is independent of r.